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Remarks on best approximation in $\mathbb{R}$-trees

Abstract. An $\mathbb{R}$-tree is a geodesic space for which there is a unique arc joining any two of its points, and this arc is a metric segment. If X is a closed convex subset of an $\mathbb{R}$-tree Y, and if $T : X \to 2^Y$ is a multivalued mapping, then a point z for which

$$0 < \operatorname{dist}(z, T(z)) = \inf_{x \in X} \operatorname{dist}(x, T(z))$$

is called a point of best approximation. It is shown here that if T is an ε-semicontinuous mapping whose values are nonempty closed convex subsets of Y, and if T has at least two distinct points of best approximation, then T must have a fixed point. We also obtain a common best approximation theorem for a commuting pair of mappings $t : X \to Y$ and $T : X \to 2^Y$ where t is single-valued continuous and T is ε-semicontinuous.

1. Introduction. In [3] the authors extended Ky Fan's well-known best approximation theorem [1] to upper semicontinuous mappings defined on a geodesically bounded $\mathbb{R}$-tree X and taking values in the family of nonempty closed convex subsets of X. In [5] J. Markin obtained the same result for 'almost lower semicontinuous' mappings. Subsequently B. Piątek [6] proved a theorem that contains both of these results by introducing a more general concept of semicontinuity. In this note we show that under Piątek's assumption, if there is more than one point of best approximation, then the mapping must have a fixed point. This can be viewed as an extension of the following elementary fact: If $[a, b]$ is a real line interval and if a continuous map $f : [a, b] \to \mathbb{R}$ satisfies $f(a) \leq a$ and $f(b) \geq b$, then f has a fixed

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point. We also include an observation about common best approximations for commuting mappings.

2. Notation and definitions.

Definition 2.1. An $\mathbb{R}$-*tree* (*or metric tree*) is a metric space X such that:
(i) there is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in X$;
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

From (i) and (ii) it is easy to deduce:
(iii) *If* $p, q, r \in X$, *then* $[p, q] \cap [p, r] = [p, w]$ *for some* $w \in X$.

We will use the notation $(x, y]$ to denote $[x, y] \setminus \{x\}$.

Let C be a subset of an $\mathbb{R}$-tree X. For $x \in X$, let

$$\operatorname{dist}(x, C) = \inf \{d(x, y) : y \in C\}.$$

By $N_\varepsilon(C)$ we will denote the set $\{x \in X : \operatorname{dist}(x, C) \leq \varepsilon\}$. $B(x; \varepsilon)$ will denote the closed ball centered at x with radius ε.

Definition 2.2. Let X and Y be metric spaces. A mapping $T : X \to 2^Y$ with nonempty values is said to be *almost lower semicontinuous at* $x \in X$ if for each $\varepsilon > 0$ there is an open neighborhood U of x such that

$$\bigcap_{u \in U} N_\varepsilon(T(u)) \neq \emptyset.$$

T is said to be *almost lower semicontinuous* if it is almost lower semicontinuous at each $x \in X$. The mapping T is said to be *upper semicontinuous at* $x \in X$ if for any neighborhood U of T(x) there is an $\varepsilon > 0$ such that $u \in B(x; \varepsilon) \Rightarrow T(u) \subseteq U$. T is said to be *upper semicontinuous* if it is upper semicontinuous at each $x \in X$.

In [6] Piątek introduces a definition of semicontinuity which includes both of the above definitions.

Definition 2.3 ([6]). Let X and Y be metric spaces. A mapping $T : X \to 2^Y$ with nonempty values is said to be ε-*semicontinuous at* $x \in X$ if for each $\varepsilon > 0$ there is an open neighborhood U of x such that

$$T(u) \cap N_\varepsilon(T(x)) \neq \emptyset$$

for all $u \in U$.

1. ([6]) Suppose T is almost lower semicontinuous at $x \in X$. Then given $\varepsilon > 0$ there exists a neighborhood U of x such that

$$N_{\varepsilon/3}(T(x)) \cap \bigcap_{u \in U} N_{\varepsilon/3}(T(u)) \neq \emptyset.$$

So for each $u \in U$ there exists $z \in N_{\varepsilon/3}(T(x)) \cap \bigcap_{u \in U} N_{\varepsilon/3}(T(u))$, $y \in T(u)$, and $y_0 \in T(x)$ such that

$$d(y, z) \leq \varepsilon/2 \quad \text{and} \quad (y_0, z) \leq \varepsilon/2.$$

Hence $d(y, T(x)) \leq d(y, z) + d(y_0, z) \leq \varepsilon$. This implies $y \in N_\varepsilon(T(x))$. Since $y \in T(u)$,

$$T(u) \cap N_\varepsilon(T(x)) \neq \emptyset.$$

2. Now suppose T is upper semicontinuous at $x \in X$ and let $\varepsilon > 0$. Then there is a neighborhood U of x such that $T(u) \subset N_\varepsilon(T(x))$ for all $u \in U$. Thus trivially

$$T(u) \cap N_\varepsilon(T(x)) \neq \emptyset$$

for all $u \in U$.

3. Main results. Our main result is the following.

Theorem 3.1. *Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree Y, and $T : X \to 2^Y$ is an ε-semicontinuous mapping whose values are nonempty closed convex subsets of Y. Suppose also that there exist distinct points $z_1, z_2 \in X$ such that $[z_i, y_i] \cap X = \{z_i\}$ for each $y_i \in T(z_i)$, $i = 1, 2$. Then T has a fixed point.*

This result can be reworded as follows.

Theorem 3.2. *Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree Y, and $T : X \to 2^Y$ is an ε-semicontinuous mapping whose values are nonempty closed convex subsets of Y. Then either T has a fixed point or there exists at most one point $z \in X$ such that*

$$0 < \operatorname{dist}(z, T(z)) = \inf_{x \in X} \operatorname{dist}(x, T(z)). \tag{1}$$

Proof. If z satisfies (1), then $(z, y] \cap X = \emptyset$ for each $y \in T(z)$. □

The above theorem, in conjunction with the result of [6] yields the following fact. The assumption that the space X is geodesically bounded means that X does not contain a geodesic of infinite length. This assumption is of course much weaker than compactness.

Theorem 3.3. *Suppose X is a closed convex geodesically bounded subset of a complete $\mathbb{R}$-tree Y, and let $T : X \to 2^Y$ be an ε-semicontinuous mapping whose values are nonempty closed convex subsets of X. Then either T has a fixed point, or there exists a unique point $z \in X$ such that*

$$0 < \operatorname{dist}(z, T(z)) = \inf_{x \in X} \operatorname{dist}(x, T(z)). \tag{2}$$

Proof. Theorem 5 of [6] assures the existence of at least one point for which $\operatorname{dist}(z, T(z)) = \inf_{x \in X} \operatorname{dist}(x, T(z))$. □

Proof of Theorem 3.1. Let S denote the unique metric segment with endpoints z_1 and z_2. For $x \in S$, let $f(x)$ denote the unique point of $T(x)$ which is nearest to x. The structure of Y assures the existence of a unique point $\xi(x) \in S$ which is nearest to $f(x)$. Clearly $\xi(z_i) = z_i$, $i = 1, 2$. Let

$$C := \{x \in S : f(x) \neq \xi(x)\}.$$

We assert that for each $x \in C$ there exists $\varepsilon_x > 0$ such that if $u \in S$ satisfies $d(u, x) < \varepsilon_x$, then $\xi(u) = \xi(x)$, and in particular $u \in C$. Indeed let $\delta_x = \operatorname{dist}(f(x), S)$ and choose $\varepsilon_x > 0$ so that $d(u, x) < \varepsilon_x$,

$$u \in X \Rightarrow T(u) \cap N_{\delta_x/2}(T(x)) \neq \emptyset.$$

Let $w \in T(u) \cap N_{\delta_x/2}(T(x))$. Since the segment $[w, f(x)]$ lies entirely in $N_{\delta_x/2}(T(x))$ it must be the case that $[w, f(x)] \cap S = \emptyset$. Now suppose $\xi(u) \neq \xi(x)$. Then the path

$$[\xi(x), f(x)] \cup [f(x), w] \cup [w, f(u)] \cup [u, \xi(u)] \cup [\xi(u), \xi(x)]$$

would form a loop in X – a contradiction. It follows that $\xi(u) = \xi(x)$.

Now let $F := \{x \in S : \xi(x) = x\}$. Any point $x \in F \backslash C$ is clearly a fixed point of T and we are finished if $F \backslash C \neq \emptyset$. So we suppose $F \subset C$ and show that this leads to a contradiction. The preceding argument shows that the set F consists of isolated points of S. By redefining z_1, z_2 if necessary, we may suppose that $F = \{z_1, z_2\}$, i.e., we may suppose that $\xi(x) \neq x$ for all $x \in [z_1, z_2]$ with $z_1 \neq x \neq z_2$.

Let

$$A := \{x \in S : \xi(x) \in [x, z_1]\};$$
$$B := \{x \in S : \xi(x) \in [x, z_2]\}.$$

We now show that A is an open subset of S. The argument at the outset shows that there is a neighborhood U of z_1 such that $\xi(u) = z_1$ for each $u \in U$. Suppose $x \in A$ with $x \neq z_1$. Then $\delta = d(x, \xi(x)) > 0$. If some neighborhood of x is in A there is nothing to prove. Otherwise we can choose a point u of S sufficiently near x so that (i) $T(u) \cap N_{\delta/2}(T(x)) \neq \emptyset$, (ii) $d(u, x) < d(x, \xi(x))$, and (iii) $u \notin A$. Let $w \in T(u) \cap N_{\delta/2}(T(x))$. Conditions (ii) and (iii) imply $\xi(u) \neq \xi(x)$. Since $u \notin T(u)$, the path

$$[\xi(u), w] \cup [w, f(x)] \cup [f(x), \xi(x)] \cup [\xi(x), \xi(u)]$$

is a loop. Therefore we conclude that A is open, and it follows similarly that B is open. Since $A \cup B = S$ we conclude that $A \cap B \neq \emptyset$. But if $x \in A \cap B$, then $\xi(x) = x$, contradicting our assumption. □

Corollary 3.4. *Suppose X is a closed convex subset of a complete geodesically bounded $\mathbb{R}$-tree Y and suppose $f : X \to Y$ is continuous. Then either f has a fixed point, or there exists a unique point $z \in X$ such that*

$$0 < d(z, f(z)) = \inf_{x \in X} d(x, f(z)).$$

4. Common best approximations. Again let X be a closed convex subset of an $\mathbb{R}$-tree Y. Two mappings $t : X \to X$ and $T : X \to 2^X$ are said to *commute* if $t(T(x)) \subset T(t(x))$ for all $x \in X$.

It is known that the nearest point projection $p : Y \to X$ is nonexpansive.

Theorem 4.1. *Let X be a closed convex geodesically bounded subset of a complete $\mathbb{R}-$tree Y. Suppose $t : X \to Y$ is a continuous mapping and $T : X \to 2^Y$ is an ε-semicontinuous mapping with nonempty closed convex values. Suppose t and T satisfy*
(1) $\mathrm{Fix}(p \circ t)$ *is a convex subset of X,*
(2) *$p \circ t$ and $p \circ T$ commute.*

Then t and T have a common best approximation, i.e., there exists $z \in X$ such that

$$d(z, t(z)) = \inf_{x \in X} d(x, t(z)) \quad \text{and}$$
$$\mathrm{dist}\,(z, T(z)) = \inf_{x \in X} \mathrm{dist}\,(x, T(z))\,.$$

Proof. The proof follows the ideas of the proofs of [3, Theorem 2.1 (p. 684)], [7, Theorem 4.1] and [4, Theorem 5.1]. Since $p : Y \to X$ is nonexpansive and $T : X \to 2^Y$ is $\varepsilon-$semicontunuous, $p \circ T : X \to 2^X$ is $\varepsilon-$semicontinuous and has a fixed point by [6, Theorem 4]. Since $p \circ t : X \to X$ is continuous, by Theorem 3.4 of [2] $\mathrm{Fix}(p \circ t) \neq \emptyset$ and it is convex by (1). It is easy to see that $\mathrm{Fix}(p \circ t)$ is closed in X. We now let $A = \mathrm{Fix}(p \circ t)$. From (2) we have

$$p \circ t(p \circ T(x)) \subset p \circ T(x) \quad \text{for all} \quad x \in A.$$

Again by [2, Theorem 3.4], $p \circ t$ has a fixed point in $p \circ T(x)$ and hence $p \circ T(x) \cap A \neq \emptyset$ for each $x \in A$. Now we define $F : A \to 2^A$ by

$$F(x) = p \circ T(x) \cap A \quad \text{for each} \quad x \in A.$$

By [6, Lemma 2], F is an ε-semicontinuous mapping. By [6, Theorem 4], F has a fixed point, i.e., there exists $z \in A$ such that $z \in p \circ T(z) \cap A$. This implies $z \in p \circ T(z)$ and $z = p \circ t(z)$. Therefore

$$d(z, t(z)) = d(p \circ t(z), t(z)) = \inf_{x \in X} d(x, t(z)).$$

For showing that $\mathrm{dist}(z, T(z)) = \inf_{x \in X} \mathrm{dist}(x, T(z))$ we separate to two cases.
Case 1. $T(z) \cap X = \emptyset$. Since both $T(z)$ and X are convex and closed, and they are disjoint it must be the case that $p \circ T(z) = \{z\}$. Hence

$$\mathrm{dist}(z, T(z)) = \mathrm{dist}(p \circ T(z), T(z)) = \inf_{x \in X} \mathrm{dist}(x, T(z)).$$

Case 2. $T(z) \cap X \neq \emptyset$. Thus $z \in p \circ T(z) = X \cap T(z)$. This implies $z \in T(z)$ and hence the conclusion follows. □

As a consequence, we obtain the following corollary.

Corollary 4.2. *Let X be a closed convex geodesically bounded subset of a complete $\mathbb{R}-$tree Y. Suppose $t : X \to Y$ is a nonexpansive mapping and $T : X \to 2^Y$ is an ε-semicontinuous mapping with nonempty closed convex values. Suppose that $p \circ t$ and $p \circ T$ commute. Then t and T have a common best approximation.*

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