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On third-order Jacobsthal numbers and their bihyperbolic generalizations

ABSTRACT. In this paper, we introduce bihyperbolic third-order Jacobsthal and third-order Jacobsthal–Lucas numbers. In other words, bihyperbolic numbers whose coefficients are consecutive third-order Jacobsthal and third-order Jacobsthal–Lucas numbers. Furthermore, we study one parameter generalizations of bihyperbolic third-order Jacobsthal and third-order Jacobsthal–Lucas numbers and relations between them.

1. Preliminary results. Let $n \geq 0$ be an integer. The n th third-order Jacobsthal number $J_n^{(3)}$ and the n th third-order Jacobsthal–Lucas number $j_n^{(3)}$ are defined recursively by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}, \quad J_0^{(3)} = 0, \quad J_1^{(3)} = J_2^{(3)} = 1$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)}, \quad j_0^{(3)} = 2, \quad j_1^{(3)} = 1, \quad j_2^{(3)} = 5,$$

respectively. Note that third-order Jacobsthal numbers are introduced by Cook and Bacon (see [6]) as a generalization of the Jacobsthal numbers (see [7]). The Binet type formula of these sequence have the form $J_n^{(3)} = \frac{1}{7} [2^{n+1} + X_n - 2X_{n+1}]$ and $j_n^{(3)} = \frac{1}{7} [2^{n+3} - 3(X_n - 2X_{n+1})]$, where X_n is

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defined recursively by

$$X_{n+2} = -X_{n+1} - X_n, \quad X_0 = 0, \quad X_1 = 1.$$

Third-order Jacobsthal sequence has been studied in many papers, see for example [8, 9, 10, 11, 14]. Recently, Morales [12] introduced the generalized third-order Jacobsthal numbers as follows. Let $k \geq 1$ be a fixed integer, let $J_{k,n}^{(3)}$ be the n th generalized third-order Jacobsthal number defined by

$$(1) \quad J_{k,n+3}^{(3)} = (k-1)J_{k,n+2}^{(3)} + (k-1)J_{k,n+1}^{(3)} + kJ_{k,n}^{(3)},$$

where $J_{k,0}^{(3)} = 0$, $J_{k,1}^{(3)} = 1$ and $J_{k,2}^{(3)} = k-1$. It is easily seen that $J_{2,n}^{(3)} = J_n^{(3)}$.

By analogy, we define the generalized third-order Jacobsthal–Lucas numbers in the following way. Let $k \geq 1$ be a fixed integer, let $j_{k,n}^{(3)}$ be the n th generalized third-order Jacobsthal–Lucas number defined by

$$(2) \quad j_{k,n+3}^{(3)} = (k-1)j_{k,n+2}^{(3)} + (k-1)j_{k,n+1}^{(3)} + kj_{k,n}^{(3)},$$

where $j_{k,0}^{(3)} = 2$, $j_{k,1}^{(3)} = k-1$ and $j_{k,2}^{(3)} = k^2+1$. Moreover, we have $j_{2,n}^{(3)} = j_n^{(3)}$.

Some properties, identities and matrix generators of $J_{k,n}^{(3)}$ were given in [12]. In the next section of the paper, we use the following results.

Theorem 1.1 ([12], Theorem 2.3). *Let $n \geq 0$, $k \geq 1$ be integers. Then*

$$(3) \quad J_{k,n}^{(3)} = \frac{1}{k^2 + k + 1} [k^{n+1} + X_n - kX_{n+1}]$$

and

$$(4) \quad j_{k,n}^{(3)} = \frac{1}{k^2 + k + 1} [(k^2 + k + 2)k^n - (k+1)(X_n - kX_{n+1})],$$

where $X_{n+2} = -X_{n+1} - X_n$, $X_0 = 0$ and $X_1 = 1$.

Theorem 1.2 ([12], Theorem 3.3). *Let $n \geq m \geq 0$, $k \geq 1$ be integers. Then*

$$(5) \quad J_{k,m+1}^{(3)}J_{k,n}^{(3)} - J_{k,m}^{(3)}J_{k,n+1}^{(3)} = \frac{1}{\sigma_k} [k^{n+1}X_{m+1} - k^{m+1}X_{n+1} + X_{n-m}]$$

and

$$(6) \quad \begin{aligned} & j_{k,m+1}^{(3)}j_{k,n}^{(3)} - j_{k,m}^{(3)}j_{k,n+1}^{(3)} \\ &= \frac{(k+1)(\sigma_k+1)}{\sigma_k} \left[k^m X_{n+1} - k^n X_{m+1} + \frac{k+1}{\sigma_k+1} X_{n-m} \right], \end{aligned}$$

where $\sigma_k = k^2 + k + 1$.

Now we prove a new result on these sequences that will be used later. This result shows the relation between two consecutive terms of these sequences and their relationship with the sequence X_n .

Proposition 1.3. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$(7) \quad J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} = X_{n+1} = -\frac{1}{k+1} \left[j_{k,n+1}^{(3)} - kj_{k,n}^{(3)} \right],$$

where X_n as in Theorem 1.1.

Proof. (By induction on n) If $n = 0$, then $J_{k,1}^{(3)} - kJ_{k,0}^{(3)} = 1 - k \cdot 0 = 1 = X_1$. Now assume that for any $n \geq 0$, we have $J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} = X_{n+1}$ and $J_{k,n+2}^{(3)} - kJ_{k,n+1}^{(3)} = X_{n+2}$. We shall show that $J_{k,n+3}^{(3)} - kJ_{k,n+2}^{(3)} = X_{n+3}$. Applying the induction's hypothesis, we obtain

$$\begin{aligned} X_{n+3} &= -X_{n+2} - X_{n+1} \\ &= -\left(J_{k,n+2}^{(3)} - kJ_{k,n+1}^{(3)} \right) - \left(J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} \right) \\ &= -J_{k,n+2}^{(3)} + (k-1)J_{k,n+1}^{(3)} + kJ_{k,n}^{(3)} \\ &= -J_{k,n+2}^{(3)} + J_{k,n+3}^{(3)} - (k-1)J_{k,n+2}^{(3)} \\ &= J_{k,n+3}^{(3)} - kJ_{k,n+2}^{(3)} \end{aligned}$$

and by induction's rule the first formula follows. The second statement is similar. □

The third-order Jacobsthal numbers and their generalizations have applications also in the theory of hypercomplex numbers. In [8], the author introduced and studied third-order Jacobsthal quaternions. In [9], the author considered the dual third-order Jacobsthal numbers and dual third-order Jacobsthal vectors. In this paper, we use third-order Jacobsthal, third-order Jacobsthal–Lucas and their generalizations in the theory of bihyperbolic numbers.

Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit (or unipotent) is an element $h \neq \pm 1$ such that $h^2 = 1$. Bihyperbolic numbers are a generalization of hyperbolic numbers. A bihyperbolic number has the form

$$\Xi = \xi_0 + \xi_1 h_1 + \xi_2 h_2 + \xi_3 h_3, \quad \xi_l \in \mathbb{R}, \quad l = 0, 1, 2, 3.$$

Note that $h_l \notin \mathbb{R}$ ($l = 1, 2, 3$) are operators such that

$$(8) \quad h_1^2 = h_2^2 = h_3^2 = 1$$

and

$$(9) \quad h_1 = h_2 h_3 = h_3 h_2, \quad h_2 = h_1 h_3 = h_3 h_1, \quad h_3 = h_1 h_2 = h_2 h_1.$$

The addition and multiplication of bihyperbolic numbers is commutative and associative, that is the set of bihyperbolic numbers is a commutative ring. For interested readers in the algebraic properties of bihyperbolic numbers, see [1] and the references cited therein. The Fibonacci numbers and

their generalizations have applications also in the theory of bihyperbolic numbers. In [2, 3], Bród, Szynal-Liana and Włoch introduced and studied bihyperbolic numbers whose coefficients are consecutive Fibonacci numbers. Other interesting studies related to this type of sequences are introduced and studied in [4, 5].

Let $n \geq 0$ be an integer. The n th bihyperbolic third-order Jacobsthal number $BhJ_n^{(3)}$ and the n th bihyperbolic third-order Jacobsthal–Lucas $Bhj_n^{(3)}$ are defined by

$$BhJ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)}h_1 + J_{n+2}^{(3)}h_2 + J_{n+3}^{(3)}h_3$$

and

$$Bhj_n^{(3)} = j_n^{(3)} + j_{n+1}^{(3)}h_1 + j_{n+2}^{(3)}h_2 + j_{n+3}^{(3)}h_3,$$

respectively, where $J_n^{(3)}$ is the n th third-order Jacobsthal number, $j_n^{(3)}$ is the n th third-order Jacobsthal–Lucas number and h_1, h_2, h_3 are units which satisfy equations (8) and (9).

Definition 1.4. The n th generalized bihyperbolic third-order Jacobsthal (or bihyperbolic third-order k -Jacobsthal) number $BhJ_{k,n}^{(3)}$ we define in the following way

$$(10) \quad BhJ_{k,n}^{(3)} = J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3,$$

where $J_{k,n}^{(3)}$ is the n th third-order k -Jacobsthal defined in (1). By analogy, the n th generalized bihyperbolic third-order Jacobsthal–Lucas (or bihyperbolic third-order k -Jacobsthal–Lucas) number $Bhj_{k,n}^{(3)}$ we define in the following way

$$(11) \quad Bhj_{k,n}^{(3)} = j_{k,n}^{(3)} + j_{k,n+1}^{(3)}h_1 + j_{k,n+2}^{(3)}h_2 + j_{k,n+3}^{(3)}h_3,$$

where $j_{k,n}^{(3)}$ is the n th third-order k -Jacobsthal–Lucas defined in (2). For $k = 2$, we obtain $BhJ_{2,n}^{(3)} = BhJ_n^{(3)}$ and $Bhj_{2,n}^{(3)} = Bhj_n^{(3)}$.

Using the above definitions, we can write initial bihyperbolic third-order k -Jacobsthal numbers

$$(12) \quad \begin{aligned} BhJ_{k,0}^{(3)} &= h_1 + (k-1)h_2 + (k^2-k)h_3, \\ BhJ_{k,1}^{(3)} &= 1 + (k-1)h_1 + (k^2-k)h_2 + (k^3-k^2+1)h_3, \\ BhJ_{k,2}^{(3)} &= k-1 + (k^2-k)h_1 + (k^3-k^2+1)h_2 + (k^4-k^3+k-1)h_3, \end{aligned}$$

bihyperbolic third-order k -Jacobsthal–Lucas numbers

$$(13) \quad \begin{aligned} Bhj_{k,0}^{(3)} &= 2 + (k-1)h_1 + (k^2+1)h_2 + (k^3+k)h_3, \\ Bhj_{k,1}^{(3)} &= k-1 + (k^2+1)h_1 + (k^3+k)h_2 + (k^4+k^2-k-1)h_3, \\ Bhj_{k,2}^{(3)} &= k^2+1 + (k^3+k)h_1 + (k^4+k^2-k-1)h_2 + (k^5+k^3-k^2+1)h_3, \end{aligned}$$

bihyperbolic third-order Jacobsthal numbers

$$\begin{aligned} BhJ_0^{(3)} &= h_1 + h_2 + 2h_3, \\ BhJ_1^{(3)} &= 1 + h_1 + 2h_2 + 5h_3, \\ BhJ_2^{(3)} &= 1 + 2h_1 + 5h_2 + 9h_3, \end{aligned}$$

bihyperbolic third-order Jacobsthal–Lucas numbers

$$\begin{aligned} Bhj_0^{(3)} &= 2 + h_1 + 5h_2 + 10h_3, \\ Bhj_1^{(3)} &= 1 + 5h_1 + 10h_2 + 17h_3, \\ Bhj_2^{(3)} &= 5 + 10h_1 + 17h_2 + 37h_3. \end{aligned}$$

2. Main results. In this section, we present some properties of bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Theorem 2.1. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$BhJ_{k,n+3}^{(3)} = (k-1)BhJ_{k,n+2}^{(3)} + (k-1)BhJ_{k,n+1}^{(3)} + kBhJ_{k,n}^{(3)},$$

where $BhJ_{k,0}^{(3)}$, $BhJ_{k,1}^{(3)}$ and $BhJ_{k,2}^{(3)}$ are defined by (12).

Proof. By formulas (1) and (10), we have

$$\begin{aligned} &(k-1)BhJ_{k,n+2}^{(3)} + (k-1)BhJ_{k,n+1}^{(3)} + kBhJ_{k,n}^{(3)} \\ &= (k-1) \left[J_{k,n+2}^{(3)} + J_{k,n+3}^{(3)}h_1 + J_{k,n+4}^{(3)}h_2 + J_{k,n+5}^{(3)}h_3 \right] \\ &\quad + (k-1) \left[J_{k,n+1}^{(3)} + J_{k,n+2}^{(3)}h_1 + J_{k,n+3}^{(3)}h_2 + J_{k,n+4}^{(3)}h_3 \right] \\ &\quad + k \left[J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3 \right] \\ &= (k-1)J_{k,n+2}^{(3)} + (k-1)J_{k,n+1}^{(3)} + kJ_{k,n}^{(3)} \\ &\quad + \left[(k-1)J_{k,n+3}^{(3)} + (k-1)J_{k,n+2}^{(3)} + kJ_{k,n+1}^{(3)} \right] h_1 \\ &\quad + \left[(k-1)J_{k,n+4}^{(3)} + (k-1)J_{k,n+3}^{(3)} + kJ_{k,n+2}^{(3)} \right] h_2 \\ &\quad + \left[(k-1)J_{k,n+5}^{(3)} + (k-1)J_{k,n+4}^{(3)} + kJ_{k,n+3}^{(3)} \right] h_3 \\ &= J_{k,n+3}^{(3)} + J_{k,n+4}^{(3)}h_1 + J_{k,n+5}^{(3)}h_2 + J_{k,n+6}^{(3)}h_3 \\ &= BhJ_{k,n+3}^{(3)}, \end{aligned}$$

which proves what was requested. \square

In the same way, using the formulas (2) and (11), we can prove the next result.

Theorem 2.2. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj_{k,n+3}^{(3)} = (k-1)Bhj_{k,n+2}^{(3)} + (k-1)Bhj_{k,n+1}^{(3)} + kBhj_{k,n}^{(3)},$$

where $Bhj_{k,0}^{(3)}$, $Bhj_{k,1}^{(3)}$ and $Bhj_{k,2}^{(3)}$ are defined by equation (13).

Now we study the comparison between two consecutive terms of bihyperbolic third-order k -Jacobsthal numbers and their relation with the sequence X_n . Note that $X_{n+2} = -X_{n+1} - X_n$ for all integer $n \geq 0$.

Theorem 2.3. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we obtain*

$$BhJ_{k,n+1}^{(3)} = kBhJ_{k,n}^{(3)} + (h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1},$$

where X_n is defined by Theorem 1.1.

Proof. Using the relation $J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} = X_{n+1}$ in Proposition 1.3 and equation (10), we get

$$\begin{aligned} BhJ_{k,n+1}^{(3)} - kBhJ_{k,n}^{(3)} &= J_{k,n+1}^{(3)} + J_{k,n+2}^{(3)}h_1 + J_{k,n+3}^{(3)}h_2 + J_{k,n+4}^{(3)}h_3 \\ &\quad - k \left[J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3 \right] \\ &= J_{k,n+1}^{(3)} - kJ_{k,n}^{(3)} + \left[J_{k,n+2}^{(3)} - kJ_{k,n+1}^{(3)} \right] h_1 \\ &\quad + \left[J_{k,n+3}^{(3)} - kJ_{k,n+2}^{(3)} \right] h_2 + \left[J_{k,n+4}^{(3)} - kJ_{k,n+3}^{(3)} \right] h_3 \\ &= X_{n+1} + X_{n+2}h_1 + X_{n+3}h_2 + X_{n+4}h_3 \\ &= X_{n+1} + (-X_{n+1} - X_n)h_1 + X_n h_2 + X_{n+1}h_3 \\ &= (h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1}, \end{aligned}$$

which proves what was requested. \square

Remark 2.4. Using the definition of sequence X_n in the above result, we have

$$(h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1} = \begin{cases} 1 - h_1 + h_3 & \text{if } n \equiv 0 \pmod{3} \\ -1 + h_2 - h_3 & \text{if } n \equiv 1 \pmod{3} \\ h_1 - h_2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Finally, we can write Theorem 2.3 as

$$BhJ_{k,n+1}^{(3)} = \begin{cases} kBhJ_{k,n}^{(3)} + 1 - h_1 + h_3 & \text{if } n \equiv 0 \pmod{3} \\ kBhJ_{k,n}^{(3)} - 1 + h_2 - h_3 & \text{if } n \equiv 1 \pmod{3} \\ kBhJ_{k,n}^{(3)} + h_1 - h_2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

In the same way, using the relation $j_{k,n+1}^{(3)} - kj_{k,n}^{(3)} = -(k+1)X_{n+1}$ in Proposition 1.3 and (11), we can prove the next result.

Theorem 2.5. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj_{k,n+1}^{(3)} = kBhj_{k,n}^{(3)} - (k+1)[(h_2 - h_1)X_n + (h_3 - h_1 + 1)X_{n+1}],$$

where X_n is defined by Theorem 1.1.

Next theorems give a kind of the Binet formulas for the bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Theorem 2.6. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$BhJ_{k,n}^{(3)} = \Xi_k J_{k,n}^{(3)} + BhJ_{k,0}^{(3)} X_{n+1} - BhJ_{k,-1}^{(3)} X_n.$$

where $\Xi_k = 1 + kh_1 + k^2h_2 + k^3h_3$ and X_n is defined by Theorem 1.1.

Proof. Using Proposition 1.3, we have

$$\begin{aligned} J_{k,n+1}^{(3)} &= kJ_{k,n}^{(3)} + X_{n+1}, \\ J_{k,n+2}^{(3)} &= kJ_{k,n+1}^{(3)} + X_{n+2} = k^2J_{k,n}^{(3)} + (k-1)X_{n+1} - X_n, \\ J_{k,n+3}^{(3)} &= kJ_{k,n+2}^{(3)} + X_{n+3} = k^3J_{k,n}^{(3)} + (k^2 - k)X_{n+1} - (k-1)X_n. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} BhJ_{k,n}^{(3)} &= J_{k,n}^{(3)} + J_{k,n+1}^{(3)}h_1 + J_{k,n+2}^{(3)}h_2 + J_{k,n+3}^{(3)}h_3 \\ &= J_{k,n}^{(3)} + [kJ_{k,n}^{(3)} + X_{n+1}]h_1 + [k^2J_{k,n}^{(3)} + (k-1)X_{n+1} - X_n]h_2 \\ &\quad + [k^3J_{k,n}^{(3)} + (k^2 - k)X_{n+1} - (k-1)X_n]h_3 \\ &= (1 + kh_1 + k^2h_2 + k^3h_3)J_{k,n}^{(3)} \\ &\quad + (h_1 + (k-1)h_2 + (k^2 - k)h_3)X_{n+1} - (h_2 + (k-1)h_3)X_n. \end{aligned}$$

From equation (12) and $J_{k,-1}^{(3)} = 0$, we get $BhJ_{k,0}^{(3)} = h_1 + (k-1)h_2 + (k^2 - k)h_3$ and $BhJ_{k,-1}^{(3)} = h_2 + (k-1)h_3$. Finally, using the notation $\Xi_k = 1 + kh_1 + k^2h_2 + k^3h_3$, we obtain the desired formula. \square

Theorem 2.7. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we have*

$$Bhj_{k,n}^{(3)} = \Xi_k j_{k,n}^{(3)} - (k+1) [BhJ_{k,0}^{(3)} X_{n+1} - BhJ_{k,-1}^{(3)} X_n].$$

where $\Xi_k = 1 + kh_1 + k^2h_2 + k^3h_3$ and X_n is defined by Theorem 1.1.

Proof. Using Proposition 1.3, equation (11) and proceeding analogously as in the proof of the previous theorem we obtain the desired formula. \square

Corollary 2.8. *Let $k = 2$. For any integer $n \geq 0$, we have*

$$BhJ_n^{(3)} = \Xi_2 J_n^{(3)} + (h_1 + h_2 + 2h_3)X_{n+1} - (h_2 + h_3)X_n$$

and

$$Bhj_n^{(3)} = \Xi_2 j_n^{(3)} - 3[(h_1 + h_2 + 2h_3)X_{n+1} - (h_2 + h_3)X_n],$$

where $\Xi_2 = 1 + 2h_1 + 4h_2 + 8h_3$ and X_n is defined by Theorem 1.1.

For simplicity of notation let $\alpha_k = BhJ_{k,-1}^{(3)}$, $\beta_k = BhJ_{k,0}^{(3)}$ and $Z_{k,n} = \beta_k X_{n+1} - \alpha_k X_n$. Using Theorems (2.6) and (2.7), we can write

$$(14) \quad BhJ_{k,n}^{(3)} = \Xi_k J_{k,n}^{(3)} + Z_{k,n}$$

and

$$(15) \quad Bhj_{k,n}^{(3)} = \Xi_k j_{k,n}^{(3)} - (k+1)Z_{k,n},$$

where $\Xi_k = 1 + kh_1 + k^2h_2 + k^3h_3$.

Using the Binet formulas in equations (14) and (15), we can derive the d'Ocagne identity for the bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Theorem 2.9. *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$Z_{k,m+1}Z_{k,n} - Z_{k,m}Z_{k,n+1} = (\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)X_{n-m}$$

and

$$\begin{aligned} BhJ_{k,m+1}^{(3)}BhJ_{k,n}^{(3)} - BhJ_{k,m}^{(3)}BhJ_{k,n+1}^{(3)} &= \frac{\Xi_k^2}{\sigma_k} [k^{n+1}X_{m+1} - k^{m+1}X_{n+1} + X_{n-m}] \\ &\quad + (\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)X_{n-m} \\ &\quad + \Xi_k [A_{k,m}X_{n+1} - B_{k,m}X_n] \\ &\quad - \Xi_k [A_{k,n}X_{m+1} - B_{k,n}X_m], \end{aligned}$$

where $\Xi_k = 1 + kh_1 + k^2h_2 + k^3h_3$, $A_{k,n} = \beta_k J_{k,n+1}^{(3)} + (\alpha_k + \beta_k)J_{k,n}^{(3)}$, $B_{k,n} = \alpha_k J_{k,n+1}^{(3)} - \beta_k J_{k,n}^{(3)}$, $J_{k,n}^{(3)}$ and X_n are defined by Theorem 1.1.

Proof. By formulas $Z_{k,n} = \beta_k X_{n+1} - \alpha_k X_n$ and $Z_{k,n+1} = -(\alpha_k + \beta_k)X_{n+1} - \beta_k X_n$, we get

$$\begin{aligned} Z_{k,m+1}Z_{k,n} - Z_{k,m}Z_{k,n+1} &= [-(\alpha_k + \beta_k)X_{m+1} - \beta_k X_m][\beta_k X_{n+1} - \alpha_k X_n] \\ &\quad - [\beta_k X_{m+1} - \alpha_k X_m][-(\alpha_k + \beta_k)X_{n+1} - \beta_k X_n] \\ &= (\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)[X_{m+1}X_n - X_m X_{n+1}] \\ &= (\alpha_k^2 + \alpha_k\beta_k + \beta_k^2)X_{n-m}. \end{aligned}$$

Using equations (14) and (5) and some algebraic calculations, we obtain

$$\begin{aligned}
 & BhJ_{k,m+1}^{(3)} BhJ_{k,n}^{(3)} - BhJ_{k,m}^{(3)} BhJ_{k,n+1}^{(3)} \\
 &= \left[\Xi_k J_{k,m+1}^{(3)} + Z_{k,m+1} \right] \left[\Xi_k J_{k,n}^{(3)} + Z_{k,n} \right] - \left[\Xi_k J_{k,m}^{(3)} + Z_{k,m} \right] \left[\Xi_k J_{k,n+1}^{(3)} + Z_{k,n+1} \right] \\
 &= \Xi_k^2 \left[J_{k,m+1}^{(3)} J_{k,n}^{(3)} - J_{k,m}^{(3)} J_{k,n+1}^{(3)} \right] + Z_{k,m+1} Z_{k,n} - Z_{k,m} Z_{k,n+1} \\
 &\quad + \Xi_k \left[J_{k,m+1}^{(3)} Z_{k,n} - J_{k,m}^{(3)} Z_{k,n+1} - J_{k,n+1}^{(3)} Z_{k,m} + J_{k,n}^{(3)} Z_{k,m+1} \right] \\
 &= \frac{\Xi_k^2}{\sigma_k} \left[k^{n+1} X_{m+1} - k^{m+1} X_{n+1} + X_{n-m} \right] \\
 &\quad + (\alpha_k^2 + \alpha_k \beta_k + \beta_k^2) X_{n-m} \\
 &\quad + \Xi_k [A_{k,m} X_{n+1} - B_{k,m} X_n] - \Xi_k [A_{k,n} X_{m+1} - B_{k,n} X_m],
 \end{aligned}$$

where $A_{k,n} = \beta_k J_{k,n+1}^{(3)} + (\alpha_k + \beta_k) J_{k,n}^{(3)}$ and $B_{k,n} = \alpha_k J_{k,n+1}^{(3)} - \beta_k J_{k,n}^{(3)}$, which completes the proof. \square

In the same way, using equations (15) and (6), we obtain the d’Ocagne identity for the bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Theorem 2.10. *Let $n \geq 0, m \geq 0$ be integers such that $n \geq m$. Then*

$$\begin{aligned}
 & Bhj_{k,m+1}^{(3)} Bhj_{k,n}^{(3)} - Bhj_{k,m}^{(3)} Bhj_{k,n+1}^{(3)} \\
 &= \frac{\Xi_k^2 (k+1)(\sigma_k+1)}{\sigma_k} \left[k^m X_{n+1} - k^n X_{m+1} + \frac{k+1}{\sigma_k+1} X_{n-m} \right] \\
 &\quad + (k+1)^2 (\alpha_k^2 + \alpha_k \beta_k + \beta_k^2) X_{n-m} \\
 &\quad + (k+1) \Xi_k [C_{k,n} X_{m+1} - D_{k,n} X_m] \\
 &\quad - (k+1) \Xi_k [C_{k,m} X_{n+1} - D_{k,m} X_n],
 \end{aligned}$$

where $C_{k,n} = \beta_k j_{k,n+1}^{(3)} + (\alpha_k + \beta_k) j_{k,n}^{(3)}$, $D_{k,n} = \alpha_k j_{k,n+1}^{(3)} - \beta_k j_{k,n}^{(3)}$, $j_{k,n}^{(3)}$ and X_n are defined by Theorem 1.1.

Now, for $m = n - 1$ and $n \geq 1$, we obtain Cassini identities for the bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Corollary 2.11. *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}
 \left[BhJ_{k,n}^{(3)} \right]^2 - BhJ_{k,n-1}^{(3)} BhJ_{k,n+1}^{(3)} &= \frac{\Xi_k^2}{\sigma_k} \left[k^{n+1} X_n - k^n X_{n+1} + 1 \right] \\
 &\quad + \alpha_k^2 + \alpha_k \beta_k + \beta_k^2 \\
 &\quad + \Xi_k [A_{k,n-1} X_{n+1} - B_{k,n-1} X_n] \\
 &\quad - \Xi_k [A_{k,n} X_n - B_{k,n} X_{n-1}]
 \end{aligned}$$

and

$$\begin{aligned}
& \left[BhJ_{k,n}^{(3)} \right]^2 - BhJ_{k,n-1}^{(3)} BhJ_{k,n+1}^{(3)} \\
&= \frac{\Xi_k^2(k+1)(\sigma_k+1)}{\sigma_k} \left[k^{n-1} X_{n+1} - k^n X_n + \frac{k+1}{\sigma_k+1} \right] \\
&+ (k+1)^2 (\alpha_k^2 + \alpha_k \beta_k + \beta_k^2) \\
&+ (k+1) \Xi_k [C_{k,n} X_n - D_{k,n} X_{n-1}] \\
&- (k+1) \Xi_k [C_{k,n-1} X_{n+1} - D_{k,n-1} X_n].
\end{aligned}$$

Now we give ordinary generating functions for the bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers.

Theorem 2.12. *Let $k \geq 1$ be a fixed integer. The generating function for the bihyperbolic third-order k -Jacobsthal number sequence $\{BhJ_{k,n}^{(3)}\}$ is given by*

$$G(\lambda) = G\left(BhJ_{k,n}^{(3)}; \lambda\right) = \frac{\left\{ \begin{array}{l} h_1 + (k-1)h_2 + (k^2-k)h_3 \\ + [1 + (k-1)h_2 + (k^2-k+1)h_3]\lambda \\ + [kh_2 + (k^2-k)h_3]\lambda^2 \end{array} \right\}}{1 - (k-1)\lambda - (k-1)\lambda^2 - k\lambda^3}.$$

Proof. Assume that the generating function of the bihyperbolic third-order k -Jacobsthal number sequence $\{BhJ_{k,n}^{(3)}\}$ has the form $G(\lambda) = BhJ_{k,0}^{(3)} + BhJ_{k,1}^{(3)}\lambda + BhJ_{k,2}^{(3)}\lambda^2 + \dots$. Then, after some algebraic calculations

$$\begin{aligned}
& [1 - (k-1)\lambda - (k-1)\lambda^2 - k\lambda^3] G(\lambda) \\
&= BhJ_{k,0}^{(3)} + BhJ_{k,1}^{(3)}\lambda + BhJ_{k,2}^{(3)}\lambda^2 + BhJ_{k,3}^{(3)}\lambda^3 \dots \\
&\quad - (k-1)BhJ_{k,0}^{(3)}\lambda - (k-1)BhJ_{k,1}^{(3)}\lambda^2 - (k-1)BhJ_{k,2}^{(3)}\lambda^3 - \dots \\
&\quad - (k-1)BhJ_{k,0}^{(3)}\lambda^2 - (k-1)BhJ_{k,1}^{(3)}\lambda^3 - \dots \\
&\quad - kBhJ_{k,0}^{(3)}\lambda^3 - \dots \\
&= BhJ_{k,0}^{(3)} + (BhJ_{k,1}^{(3)} - (k-1)BhJ_{k,0}^{(3)})\lambda \\
&\quad + (BhJ_{k,2}^{(3)} - (k-1)BhJ_{k,1}^{(3)} - (k-1)BhJ_{k,0}^{(3)})\lambda^2,
\end{aligned}$$

since $BhJ_{k,n+3}^{(3)} = (k-1)BhJ_{k,n+2}^{(3)} + (k-1)BhJ_{k,n+1}^{(3)} - kBhJ_{k,n}^{(3)}$ and the coefficients of λ^n for $n \geq 3$ are equal to zero. Furthermore, we can write $BhJ_{k,0}^{(3)} = h_1 + (k-1)h_2 + (k^2-k)h_3$, $BhJ_{k,1}^{(3)} - (k-1)BhJ_{k,0}^{(3)} = 1 + (k-1)h_2 + (k^2-k+1)h_3$ and $BhJ_{k,2}^{(3)} - (k-1)BhJ_{k,1}^{(3)} - (k-1)BhJ_{k,0}^{(3)} = kh_2 + (k^2-k)h_3$, which completes the proof. \square

Theorem 2.13. *Let $k \geq 1$ be a fixed integer. The generating function for the bihyperbolic third-order k -Jacobsthal–Lucas number sequence $\{Bhj_{k,n}^{(3)}\}$ is given by*

$$G\left(Bhj_{k,n}^{(3)}; \lambda\right) = \frac{\left\{ \begin{array}{l} 2 + (k-1)h_1 + (k^2+1)h_2 + (k^3+k)h_3 \\ +[-k+1+2kh_1+(k^2+1)h_2+(k^1-1)h_3]\lambda \\ +[2+2kh_1+(k^2-k)h_2+(k^3+k)h_3]\lambda^2 \end{array} \right\}}{1 - (k-1)\lambda - (k-1)\lambda^2 - k\lambda^3}.$$

Proof. This proof is similar to the proof of Theorem 2.12. In this case, we have $Bhj_{k,0}^{(3)} = 2 + (k-1)h_1 + (k^2+1)h_2 + (k^3+k)h_3$, $Bhj_{k,1}^{(3)} - (k-1)Bhj_{k,0}^{(3)} = -k+1+2kh_1+(k^2+1)h_2+(k^3-1)h_3$ and $Bhj_{k,2}^{(3)} - (k-1)Bhj_{k,1}^{(3)} - (k-1)Bhj_{k,0}^{(3)} = 2+2kh_1+(k^2-k)h_2+(k^3+k)h_3$. \square

Remark 2.14. Let $k = 2$, the generating function $G(BhJ_n^{(3)}; \lambda)$ for the bihyperbolic third-order Jacobsthal number sequence $\{BhJ_n^{(3)}\}_{n \geq 0}$ is

$$G\left(BhJ_n^{(3)}; \lambda\right) = \frac{h_1 + h_2 + 2h_3 + [1 + h_2 + 3h_3]\lambda + [2h_2 + 2h_3]\lambda^2}{1 - \lambda - \lambda^2 - 2\lambda^3}.$$

Also, the generating function $G(Bhj_n^{(3)}; \lambda)$ for the bihyperbolic third-order Jacobsthal–Lucas number sequence $\{Bhj_n^{(3)}\}_{n \geq 0}$ is

$$G\left(Bhj_n^{(3)}; \lambda\right) = \frac{\left\{ \begin{array}{l} 2 + h_1 + 5h_2 + 10h_3 \\ +[-1 + 4h_1 + 5h_2 + 7h_3]\lambda + [2 + 4h_1 + 2h_2 + 10h_3]\lambda^2 \end{array} \right\}}{1 - \lambda - \lambda^2 - 2\lambda^3}.$$

Finally, we will give a matrix representation of the bihyperbolic numbers defined above. Let us consider the following matrix

$$N_k^{(3)} = \begin{bmatrix} BhJ_{k,1}^{(3)} & BhJ_{k,2}^{(3)} - (k-1)BhJ_{k,1}^{(3)} & kBhJ_{k,0}^{(3)} \\ BhJ_{k,0}^{(3)} & BhJ_{k,1}^{(3)} - (k-1)BhJ_{k,0}^{(3)} & kBhJ_{k,-1}^{(3)} \\ BhJ_{k,-1}^{(3)} & BhJ_{k,0}^{(3)} - (k-1)BhJ_{k,-1}^{(3)} & kBhJ_{k,-2}^{(3)} \end{bmatrix},$$

where $BhJ_{k,-1}^{(3)} = h_2 + (k-1)h_3$ and $BhJ_{k,-2}^{(3)} = \frac{1}{k} + h_3$.

It is easy to see that for $n \geq 0$ it holds

$$\left[M_k^{(3)}\right]^n = \begin{bmatrix} k-1 & k-1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{k,n+1}^{(3)} & TJ_{k,n+1}^{(3)} & kJ_{k,n}^{(3)} \\ J_{k,n}^{(3)} & TJ_{k,n}^{(3)} & kJ_{k,n-1}^{(3)} \\ J_{k,n-1}^{(3)} & TJ_{k,n-1}^{(3)} & kJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $TJ_{k,n}^{(3)} = J_{k,n+1}^{(3)} - (k-1)J_{k,n}^{(3)}$, $J_{k,-1}^{(3)} = 0$ and $J_{k,-2}^{(3)} = \frac{1}{k}$. Then, we consider the following theorem.

Theorem 2.15. *Let $k \geq 1$ be a fixed integer. For any integer $n \geq 0$, we obtain*

$$N_k^{(3)} \cdot \left[M_k^{(3)} \right]^n = \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $ThJ_{k,n}^{(3)} = BhJ_{k,n+1}^{(3)} - (k-1)BhJ_{k,n}^{(3)}$.

Proof. (By induction on n) If $n = 0$, then assuming that the matrix to power 0 is the identity matrix the result is obvious. Now, suppose that for $n \geq 0$ it holds

$$N_k^{(3)} \cdot \left[M_k^{(3)} \right]^n = \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix}.$$

By simple calculations, using induction's hypothesis we have

$$\begin{aligned} N_k^{(3)} \cdot \left[M_k^{(3)} \right]^{n+1} &= N_k^{(3)} \cdot \left[M_k^{(3)} \right]^n \cdot M_k^{(3)} \\ &= \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix} \cdot \begin{bmatrix} k-1 & k-1 & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (k-1)BhJ_{k,n+1}^{(3)} + ThJ_{k,n+1}^{(3)} & ThJ_{k,n+2}^{(3)} & kBhJ_{k,n+1}^{(3)} \\ (k-1)BhJ_{k,n}^{(3)} + ThJ_{k,n}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ (k-1)BhJ_{k,n-1}^{(3)} + ThJ_{k,n-1}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \end{bmatrix} \\ &= \begin{bmatrix} BhJ_{k,n+2}^{(3)} & ThJ_{k,n+2}^{(3)} & kBhJ_{k,n+1}^{(3)} \\ BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \end{bmatrix}, \end{aligned}$$

where $ThJ_{k,n}^{(3)} = BhJ_{k,n+1}^{(3)} - (k-1)BhJ_{k,n}^{(3)}$, which completes the proof. \square

In addition, we can also verify the behavior of the following determinants indicated by

$$\det \left[\left(M_k^{(3)} \right)^n \right] = \det \begin{bmatrix} J_{k,n+1}^{(3)} & TJ_{k,n+1}^{(3)} & kJ_{k,n}^{(3)} \\ J_{k,n}^{(3)} & TJ_{k,n}^{(3)} & kJ_{k,n-1}^{(3)} \\ J_{k,n-1}^{(3)} & TJ_{k,n-1}^{(3)} & kJ_{k,n-2}^{(3)} \end{bmatrix},$$

$$\det \left[N_k^{(3)} \cdot \left(M_k^{(3)} \right)^n \right] = \det \begin{bmatrix} BhJ_{k,n+1}^{(3)} & ThJ_{k,n+1}^{(3)} & kBhJ_{k,n}^{(3)} \\ BhJ_{k,n}^{(3)} & ThJ_{k,n}^{(3)} & kBhJ_{k,n-1}^{(3)} \\ BhJ_{k,n-1}^{(3)} & ThJ_{k,n-1}^{(3)} & kBhJ_{k,n-2}^{(3)} \end{bmatrix},$$

where $TJ_{k,n}^{(3)}$ and $ThJ_{k,n}^{(3)}$ as in Theorem 2.15.

Corollary 2.16. *For any integer $n \geq 0$, we obtain*

$$(16) \quad \det \left[\left(M_k^{(3)} \right)^n \right] = k^n,$$

$$(17) \quad \det \left[N_k^{(3)} \cdot \left(M_k^{(3)} \right)^n \right] = \det \left[\left(M_k^{(3)} \right)^n \cdot N_k^{(3)} \right] = k^n \det \left[N_k^{(3)} \right].$$

Remark 2.17. Note that multiplication of bihyperbolic numbers is commutative and determinant properties can be used. In this sense, calculating determinants in Theorem 2.15, we can also obtain a cubic identity for the bihyperbolic third-order k -Jacobsthal and bihyperbolic third-order k -Jacobsthal–Lucas numbers. Also, using algebraic operations and matrix algebra could give many other interesting identities of these bihyperbolic numbers.

3. Conclusions. In this paper, we introduce one-parameter generalization of bihyperbolic third-order Jacobsthal numbers and bihyperbolic third-order Jacobsthal–Lucas numbers. Furthermore, we study some properties of them, among others the Binet formula, Cassini and d’Ocagne identities. Moreover, we give the generating function and special relations between them. The presented results are generalizations of the dual and gaussian third-order Jacobsthal numbers studied in [9, 13] following the ideas of Bród, Szynal-Liana and Włoch in [2, 3]. An interested reader could further generalize this sequence, considering for example arbitrary initial conditions.

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