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The existence of solutions to the nonhomogeneous degenerate nonlinear elliptic equations

Abstract. In this paper we are interested in the existence and uniqueness of solutions for Dirichlet problem associated with the degenerate nonlinear elliptic equations

$$
\begin{aligned}
&- \operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\omega_2(x) + \mathcal{B}(x,\nabla u)\,\nu_1(x)\big] + \mathcal{H}(x,u,\nabla u)\nu_2 + |u|^{p-2}u\,\omega_1\\
&= \rho_0 - \sum_{j=1}^{n} D_j\rho_j,\\
&u-\psi \in W_0^{1,p}(\Omega,\omega_1,\omega_2),
\end{aligned}
$$

in the setting of the weighted Sobolev spaces.

1. Introduction. In this paper we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ (see Definition 2.2) for the Dirichlet problem

$$
(P)\begin{cases}
-\operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\omega_2 + \mathcal{B}(x,\nabla u)\,\nu_1\big] + \mathcal{H}(x,u,\nabla u)\,\nu_2 + |u|^{p-2}u\,\omega_1\\
= \rho_0 - \displaystyle\sum_{j=1}^{n} D_j\rho_j,\\
u-\psi \in W_0^{1,p}(\Omega,\omega_1,\omega_2),
\end{cases}
$$

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where $D_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, ν_1 and ν_2, are four weight functions, $\psi \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and the functions, $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Caratheodory functions which satisfy the following conditions:
(H1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^n$,
$\xi \mapsto \mathcal{A}(x, \xi)$ is continuous on $\mathbb{R}^n$ for almost all $x \in \Omega$.
(H2) $(\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi'))\cdot(\xi - \xi')) > 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$ and $\mathcal{A}(x, \xi) = (\mathcal{A}_1(x, \xi), \ldots, \mathcal{A}_n(x, \xi))$ (where a dot denotes here the Euclidian scalar product in $\mathbb{R}^n$).
(H3) $\mathcal{A}(x, \xi)\cdot\xi \geq \lambda_1|\xi|^p$, where λ_1 is a positive constant and $1 < p < \infty$.
(H4) $|\mathcal{A}(x, \xi)| \leq h_1(x)|\xi|^{p/p'}$, where h_1 is a nonnegative function and $h_1 \in L^\infty(\Omega)$ (with $1/p + 1/p' = 1$).
(H5) $x \mapsto \mathcal{B}(x, \xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^n$,
$\xi \mapsto \mathcal{B}(x, \xi)$ is continuous on $\mathbb{R}^n$ for almost all $x \in \Omega$.
(H6) $(\mathcal{B}(x, \xi) - \mathcal{B}(x, \xi'))\cdot(\xi - \xi')\rangle \geq 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$ and $\mathcal{B}(x, \xi) = (\mathcal{B}_1(x, \xi), \ldots, \mathcal{B}_n(x, \xi))$.
(H7) $\mathcal{B}(x, \xi)\cdot\xi \geq \lambda_2|\xi|^q$, where $\lambda_2 > 0$ is a constant and $1 < q < \infty$.
(H8) $|\mathcal{B}(x, \xi)| \leq h_2(x)|\xi|^{q/q'}$, $1 < q < \infty$, h_2 is a nonnegative function and $h_2 \in L^\infty(\Omega)$, $1/q + 1/q' = 1$.
(H9) $x \mapsto \mathcal{H}(x, \eta, \xi)$ is measurable on Ω for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$,
$(\eta, \xi) \mapsto \mathcal{H}(x, \eta, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.
(H10) $[\mathcal{H}(x, \eta, \xi) - \mathcal{H}(x, \eta', \xi')](\eta - \eta') > 0$, whenever $\eta, \eta' \in \mathbb{R}$, $\eta \neq \eta'$.
(H11) $\mathcal{H}(x, \eta, \xi)\eta \geq \lambda_3|\xi|^s + \Lambda_3|\eta|^s$, where λ_3 and Λ_3 are nonnegative constants and $1 < s < \infty$.
(H12) $|\mathcal{H}(x, \eta, \xi)| \leq h_3(x)|\eta|^{s/s'} + h_4(x)|\xi|^{s/s'}$, where h_3 and h_4 are nonnegative functions, with h_3 and $h_4 \in L^\infty(\Omega)$, $1/s + 1/s' = 1$.

Let Ω be a bounded open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω, positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E) = \int_E \omega(x)\, dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [4], [5], [8] and [9]). In various applications we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaciology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [2] and [7]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [17]). These classes have found many useful applications in harmonic analysis (see [18]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [14]). There are, in fact, many interesting examples of weights (see [12] for p-admissible weights).

The following theorem will be proved in Section 3.

Theorem 1.1. *Assume* (H1)–(H12) *and* $\omega_2 \leq \omega_1$.
(i) Let $1 < q, s < p < \infty$, ω_1 *and* ω_2 *be* A_p*-weights,* $\nu_1, \nu_2 \in \mathcal{W}(\Omega)$, $\frac{\nu_1}{\omega_2} \in L^{r_1}(\Omega, \omega_2)$ *(where* $r_1 = p/(p-q)$*),* $\frac{\nu_2}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ *and* $\frac{\nu_2}{\omega_2} \in L^{r_2}(\Omega, \omega_2)$ *(where* $r_2 = p/(p-s)$*).*
(ii) $\frac{\rho_0}{\omega_1} \in L^{p'}(\Omega, \omega_1)$, $\frac{\rho_j}{\omega_2} \in L^{p'}(\Omega, \omega_2)$ *(*$j = 1, \ldots, n$*) and* $\psi \in W^{1,p}(\Omega, \omega_1, \omega_2)$.

Then the problem (P) has a unique solution $u \in W^{1,p}(\Omega, \omega_1, \omega_2)$ *with* $u - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

The paper is organized as follows. In Section 2 we present the definitions and basics results. In Section 3 we prove our main result about existence and uniqueness of solutions for problem (P).

2. Definitions and basic results. We recall some standards notations, properties and results which will be used throughout the paper.

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 \leq p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ (called A_p-constant) such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C, \quad \text{when } 1 < p < \infty,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) \leq C \operatorname*{ess\,inf}_B \omega, \quad \text{when } p = 1,$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [11], [12] or [18] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu_\omega(B(x; 2r)) \leq C\mu_\omega(B(x; r))$, for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu_\omega(B) = \int_B \omega(x)\, dx$. If $\omega \in A_p$, then μ_ω is doubling (see Corollary 15.7 in [12]).

As an example of A_p-weight one can take the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$ which is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [18]). Another example is $\omega(x) = |x|^\alpha(\max\{1, -\ln(|x|)\})^\beta$. This is an A_1-weight if and only if $-n < \alpha < 0$ or $\alpha = 0 \leq \beta$ (see Proposition 7.2 in [1]).

If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu_\omega(E)}{\mu_\omega(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu_\omega(E) = 0$ then $|E| = 0$. The measure μ_ω and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu_\omega(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e.

In order to discuss the problem (P), we need some elementary results for weighted Lebesgue spaces $L^p(\Omega,\omega)$ and the weighted Sobolev spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$ and $W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Definition 2.1. Let ω be a weight and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $1 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p\omega\,dx\right)^{1/p} < \infty.$$

We define $L^p(\Omega,\omega;\mathbb{R}^n) = \left\{\varphi : \Omega \to \mathbb{R}^n : \int_\Omega |\varphi|^p\omega\,dx < \infty\right\}$. Denote the norm of $L^p(\Omega,\omega)$ and $L^p(\Omega,\omega;\mathbb{R}^n)$ by $\|\cdot\|_{L^p(\Omega,\omega)}$.

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and $L^p(\Omega,\omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [19]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let ω_1 and ω_2 be A_p-weights ($1 < p < \infty$). We define the weighted Sobolev space $W^{1,p}(\Omega,\omega_1,\omega_2)$ as the set of functions $u \in L^p(\Omega,\omega_1)$ with weak derivatives $D_ju \in L^p(\Omega,\omega_2)$ (or $\nabla u = (D_1u,\ldots,D_nu) \in L^p(\Omega,\omega_2;\mathbb{R}^n)$, ∇u is the weak gradient of u). The norm of u in $W^{1,p}(\Omega,\omega_1,\omega_2)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega_1,\omega_2)} = \left(\int_\Omega |u|^p\omega_1\,dx + \int_\Omega |\nabla u|^p\omega_2\,dx\right)^{1/p}. \tag{2.1}$$

The space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a reflexive Banach space (see [8], [15], [16] or [22] for more information about the spaces $W^{1,p}(\Omega,\omega_1,\omega_2)$). The dual of the space $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is the space

$$[W_0^{1,p}(\Omega,\omega_1,\omega_2)]^* = \Big\{T = f_0 - \operatorname{div}(F),\ F = (f_1,\ldots,f_n) : \frac{f_0}{\omega_1} \in L^{p'}(\Omega,\omega_1),\ \frac{f_j}{\omega_2} \in L^{p'}(\Omega,\omega_2),\ j = 1,\ldots,n\Big\}.$$

In this paper we use the following result.

Theorem 2.1. *Let $\omega \in A_p$, $1 < p < \infty$ and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, \omega)$, then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$ a.e. on Ω;
(ii) $|u_{m_k}(x)| \leq \Phi(x)$ a.e. on Ω.

Proof. The proof of this theorem follows the lines of the proof of Theorem 2.8.1 in [10]. □

Definition 2.3. We say that an element $u \in W^{1,p}(\Omega, \omega_1, \omega_2)$ is a (weak) solution of equation

$$- \operatorname{div}[\mathcal{A}(x, \nabla u(x))\omega_2(x) + \mathcal{B}(x, \nabla u(x))\nu_1(x)] + \mathcal{H}(x, u, \nabla u)\nu_2 + |u|^{p-2}u\omega_1$$
$$= \rho_0 - \sum_{j=1}^{n} D_j\rho_j,$$

if

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi\, \omega_2\, dx + \int_\Omega \mathcal{B}(x, \nabla u) \cdot \nabla\varphi\, \nu_1\, dx + \int_\Omega \mathcal{H}(x, u, \nabla u)\varphi\, \nu_2\, dx$$
$$+ \int_\Omega |u|^{p-2}u\, \varphi\, \omega_1\, dx = \int_\Omega \rho_0\varphi\, dx + \sum_{j=1}^{n} \rho_j D_j\varphi\, dx$$

for all $\varphi \in W^{1,p}(\Omega, \omega_1, \omega_2)$, or we can write

$$\int_\Omega \left(\mathcal{H}(x, u, \nabla u)\varphi\frac{\nu_2}{\omega_1} + |u|^{p-2}u\, \varphi - \frac{\rho_0}{\omega_1}\varphi\right)\omega_1\, dx$$
$$+ \int_\Omega \left(\mathcal{A}(x, \nabla u) \cdot \nabla\varphi + \mathcal{B}(x, \nabla u) \cdot \nabla\varphi\frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2}\cdot\nabla\varphi\right)\omega_2\, dx = 0,$$

where $\rho = (\rho_1, \ldots, \rho_n)$.

Remark 2.2. (i) If $\frac{\nu_1}{\omega_2} \in L^{r_1}(\Omega, \omega_2)$ (where $r_1 = p/(p-q)$), then $\|u\|_{L^q(\Omega,\nu_1)} \leq C_{1,2}\|u\|_{L^p(\Omega,\omega_2)}$, where $C_{1,2} = \|\nu_1/\omega_2\|^{1/q}_{L^{r_1}(\Omega,\omega_2)}$. In fact, by Hölder's inequality we obtain

$$\begin{aligned}
\|u\|^q_{L^q(\Omega,\nu_1)} &= \int_\Omega |u|^q\nu_1\, dx \\
&= \int_\Omega |u|^q\frac{\nu_1}{\omega_2}\omega_2\, dx \\
&\leq \left(\int_\Omega |u|^{q\,(p/q)}\omega_2\, dx\right)^{q/p}\left(\int_\Omega \big(\nu_1/\omega_2\big)^{p/(p-q)}\omega_2\, dx\right)^{(p-q)/q} \\
&= \|u\|^q_{L^p(\Omega,\omega_2)}\|\nu_1/\omega_2\|_{L^{r_1}(\Omega,\omega_2)}.
\end{aligned}$$

(ii) Analogously, if $\frac{\nu_2}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$ and $\frac{\nu_2}{\omega_2} \in L^{r_2}(\Omega,\omega_2)$ (where $r_2 = p/(p-s)$), then

$$\|u\|_{L^s(\Omega,\nu_2)} \leq C_{2,1}\|u\|_{L^p(\Omega,\omega_1)},$$
$$\|u\|_{L^s(\Omega,\nu_2)} \leq C_{2,2}\|u\|_{L^p(\Omega,\omega_2)},$$

where $C_{2,1} = \|\nu_2/\omega_1\|^{1/s}_{L^{r_2}(\Omega,\omega_1)}$ and $C_{2,2} = \|\nu_2/\omega_2\|^{1/s}_{L^{r_2}(\Omega,\omega_2)}$.

Proposition 2.3. *Let $1 < p < \infty$. There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$,*

$$\beta_p(|x|+|y|)^{p-2}|x-y|^2 \leq (|x|^{p-2}x-|y|^{p-2}y)\cdot(x-y) \leq \gamma_p(|x|+|y|)^{p-2}|x-y|.$$

Proof. See Proposition 17.3 in [6]. □

In the proof of Theorem 1.1 we will use the following result.

Let X be a reflexive Banach space and denote its dual by X^*. Let $\|\cdot\|_X$ be the norm of X and $\langle\cdot\,,\cdot\rangle$ be a pairing between X and X^*.

Theorem 2.4. *Let $\mathcal{K}$ be a nonempty closed convex subset of X and let $\mathcal{T} : \mathcal{K} \to X^*$ be a monotone, coercive and weakly continuous on $\mathcal{K}$. Then there exists an element $u \in \mathcal{K}$ such that $\langle \mathcal{T}u, v-u\rangle \geq 0$ whenever $v \in \mathcal{K}$.*

Proof. See Corollary III.1.8 in [13]. □

For more information on the theory of monotone operators see [21].

3. Main result. Let $X = L^p(\Omega,\omega_1) \times L^p(\Omega,\omega_2;\mathbb{R}^n)$. The norm of X is

$$\|(g,f)\|_X = \|g\|_{L^p(\Omega,\omega_1)} + \|f\|_{L^p(\Omega,\omega_2;\mathbb{R}^n)},$$

for each element $(g,f) \in X$ ($f = (f_1,\ldots,f_n)$). Then X is a reflexive Banach space and its dual $X^* = L^{p'}(\Omega,\omega_1) \times L^{p'}(\Omega,\omega_2;\mathbb{R}^n)$. Let $\langle\cdot\,,\cdot\rangle$ be the usual pairing between X and X^*:

$$\langle (g_1,f),(g_2,h)\rangle = \int_\Omega g_1 g_2 \omega_1\,dx + \int_\Omega f\cdot h\omega_2\,dx,$$

(where $f = (f_1,\ldots,f_n)$ and $h = (h_1,\ldots,h_n)$).

Let $\psi \in W^{1,p}(\Omega,\omega_1,\omega_2)$. Define the set

$$\mathcal{K}_\psi = \{(g,\nabla g) : g \in W^{1,p}(\Omega,\omega_1,\omega_2) \text{ and } g-\psi \in W^{1,p}_0(\Omega,\omega_1,\omega_2)\}.$$

Lemma 3.1. *$\mathcal{K}_\psi$ is a nonempty closed convex subset of X.*

Proof. (a) Suppose that $(u,\nabla u) \in \mathcal{K}_\psi$. Hence $u \in W^{1,p}(\Omega,\omega_1,\omega_2)$. Then $u \in L^p(\Omega,\omega_1)$ and $\nabla u \in L^p(\Omega,\omega_2;\mathbb{R}^n)$. Therefore, $(u,\nabla u) \in X$. Thus, $\mathcal{K}_\psi \subset X$.

(b) If $(u_k, \nabla u_k) \in \mathcal{K}_\psi$ is a sequence which converges to $(g, f) \in X$ ($u_k \to g$ in $L^p(\Omega, \omega_1)$ and $\nabla u_k \to f$ in $L^p(\Omega, \omega_2)$, where $f = (f_1, \ldots, f_n) \in L^p(\Omega, \omega_2; \mathbb{R}^n)$), then (since $\omega_2 \le \omega_1$)

$$\begin{aligned}
(i)\quad & \int_\Omega |u_k - g|^p \omega_2\, dx = \int_\Omega |(u_k - \psi) - (g - \psi)|^p \omega_2\, dx \\
& \qquad \le \int_\Omega |(u_k - \psi) - (g - \psi)|^p\, \omega_1\, dx = \int_\Omega |u_k - g|^p\, \omega_1\, dx \to 0, \\
(ii)\quad & \int_\Omega |(\nabla u_k - \nabla\psi) - (f - \nabla\psi)|^p\, \omega_2\, dx = \int_\Omega |\nabla u_k - f|^p\, \omega_2\, dx \to 0,
\end{aligned}$$

as $k \to \infty$. Since $\omega_2 \in A_p$, then $\nabla g = f \in L^p(\Omega, \omega_2, \mathbb{R}^n)$ (by the uniqueness of the gradient). And since $u_k - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, then $g - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Hence $g \in W^{1,p}(\Omega, \omega_1, \omega_2)$. Therefore, $(g, f) = (g, \nabla g) \in \mathcal{K}_\psi$. Thus, $\mathcal{K}_\psi$ is closed in X.

(c) Let $(u, \nabla u), (v, \nabla v) \in \mathcal{K}_\psi$ and $\alpha \in [0, 1]$. Then $\alpha\, u + (1 - \alpha)v \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and

$$\alpha u + (1 - \alpha)v - \psi = \alpha(u - \psi) + (1 - \alpha)(v - \psi) \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

Hence $\alpha(u, \nabla u) + (1 - \alpha)(v, \nabla v) = \big(\alpha u + (1 - \alpha)v, \nabla(\alpha u + (1 - \alpha)v)\big) \in \mathcal{K}_\psi$. Therefore, $\mathcal{K}_\psi$ is convex in X. □

Now, define a mapping $\mathcal{T} : \mathcal{K}_\psi \to X^*$ by the formula

$$\begin{aligned}
\mathcal{T}(u, \nabla u) = \Bigg(& \mathcal{H}(x, u, \nabla u)\frac{\nu_2}{\omega_1} + |u|^{p-2}u - \frac{\rho_0}{\omega_1}, \\
& \qquad\qquad \mathcal{A}(x, \nabla u) + \mathcal{B}(x, \nabla u)\frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2}\Bigg),
\end{aligned} \tag{3.1}$$

where $\rho = (\rho_1, \ldots, \rho_n)$. For convenience, we denote $\mathcal{T}(u, \nabla u)$ simply by $\mathcal{T}(u)$. For each element $(g, f) \in X$, we have

$$\begin{aligned}
\langle \mathcal{T}(u), (g, f)\rangle &= \int_\Omega \left(\mathcal{H}(x, u, \nabla u)\frac{\nu_2}{\omega_1} + |u|^{p-2}u - \frac{\rho_0}{\omega_1}\right) g\, \omega_1\, dx \\
&\quad + \int_\Omega \left(\mathcal{A}(x, \nabla u) + \mathcal{B}(u, \nabla u)\frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2}\right)\cdot f\, \omega_2\, dx \\
&= \int_\Omega \mathcal{H}(x, u, \nabla u)\, g\, \nu_2\, dx + \int_\Omega |u|^{p-2}u\, g\, \omega_1\, dx - \int_\Omega \rho_0\, g\, dx \\
&\quad + \int_\Omega \mathcal{A}(x, \nabla u)\cdot f\, \omega_2\, dx + \int_\Omega \mathcal{B}(x, \nabla u)\cdot f\, \nu_1\, dx - \int_\Omega \rho\cdot f\, dx.
\end{aligned}$$

(i) By assumption (H4) we have

$$\begin{aligned}
\left|\int_\Omega \mathcal{A}(x,\nabla u)\cdot f\,\omega_2\,dx\right| &\le \int_\Omega |\mathcal{A}(u,\nabla u)|\,|f|\,\omega_2\,dx \\
&\le \int_\Omega h_1|\nabla u|^{p-1}|f|\,\omega_2\,dx \\
&\le \|h_1\|_{L^\infty(\Omega)}\|\nabla u\|^{p-1}_{L^p(\Omega,\omega_2)}\|f\|_{L^p(\Omega,\omega_2)} \\
&\le \|h_1\|_{L^\infty(\Omega)}\|(u,\nabla u)\|^{p-1}_X\|(g,f)\|_X.
\end{aligned} \tag{3.2}$$

(ii) By assumption (H8) and Remark 2.2(i) we obtain

$$\begin{aligned}
&\left|\int_\Omega \mathcal{B}(x,\nabla u)\cdot f\,\nu_1\,dx\right| \le \int_\Omega |\mathcal{B}(x,\nabla u)|\,|f|\,\nu_1\,dx \\
&\quad\le \int_\Omega h_2|\nabla u|^{q-1}|f|\,\nu_1\,dx \\
&\quad\le \|h_2\|_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{(q-1)q'}\nu_1\,dx\right)^{1/q'}\left(\int_\Omega |f|\nu_1\,dx\right)^{1/q} \\
&\quad= \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{q-1}_{L^q(\Omega,\nu_1)}\|f\|_{L^q(\Omega,\nu_1)} \\
&\quad\le \|h_2\|_{L^\infty(\Omega)}C_{1,2}^{q-1}\|\nabla u\|^{q-1}_{L^p(\Omega,\omega_2)}C_{1,2}\|f\|_{L^p(\Omega,\omega_2)} \\
&\quad\le C_{1,2}^q\|h_2\|_{L^\infty(\Omega)}\|(u,\nabla u)\|^{q-1}_X\|(g,f)\|_X.
\end{aligned} \tag{3.3}$$

(iii) By (H12) and Remark 2.2(ii) we get

$$\begin{aligned}
&\left|\int_\Omega \mathcal{H}(x,u,\nabla u)g\nu_2\,dx\right| \le \int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|g|\,\nu_2\,dx \\
&\quad\le \int_\Omega \Big(h_3|u|^{s-1}+h_4|\nabla u|^{s-1}\Big)|g|\,\nu_2\,dx \\
&\quad\le \|h_3\|_{L^\infty(\Omega)}\|u\|^{s-1}_{L^s(\Omega,\nu_2)}\|g\|_{L^s(\Omega,\nu_2)} \\
&\qquad + \|h_4\|_{L^\infty(\Omega)}\|\nabla u\|^{s-1}_{L^s(\Omega,\nu_2)}\|g\|_{L^s(\Omega,\nu_2)} \\
&\quad\le \|h_3\|_{L^\infty(\Omega)}C_{2,1}^{s-1}\|u\|^{s-1}_{L^p(\Omega,\omega_1)}C_{2,1}\|g\|_{L^p(\Omega,\omega_1)} \\
&\qquad + \|h_4\|_{L^\infty(\Omega)}C_{2,2}^{s-1}\|\nabla u\|^{s-1}_{L^p(\Omega,\omega_2)}C_{2,1}\|g\|_{L^p(\Omega,\omega_1)} \\
&\quad\le C\,\|(u,\nabla u)\|^{s-1}_X\|(g,f)\|_X,
\end{aligned} \tag{3.4}$$

where $C=\max\{C_{2,1}^s\|h_3\|_{L^\infty(\Omega)}, C_{2,1}C_{2,2}^{s-1}\|h_4\|_{L^\infty(\Omega)}\}$.

(iv) We also have

$$\left|\int_\Omega |u|^{p-2}u\, g\, \omega_1\, dx\right| \le \int_\Omega |u|^{p-1}|g|\, \omega_1\, dx \le \|u\|^{p-1}_{L^p(\Omega,\omega_1)}\|g\|_{L^p(\Omega,\omega_1)} \le \|(u,\nabla u)\|_X^{p-1}\|(g,f)\|_X. \tag{3.5}$$

(v) Moreover,

$$\begin{aligned}
&\left|\int_\Omega \rho_0\, g\, dx + \int_\Omega \rho\cdot f\, dx\right| \\
&\quad\le \int_\Omega \frac{|\rho_0|}{\omega_1}\, g\, \omega_1\, dx + \int_\Omega \frac{|\rho|}{\omega_2}|f|\, \omega_2\, dx \\
&\quad\le \|\rho_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\|g\|_{L^p(\Omega,\omega_1)} + \|\rho/\omega_2\|_{L^{p'}(\Omega,\omega_2)}\|f\|_{L^p(\Omega,\omega_2)} \\
&\quad\le \big(\|\rho_0/\omega_1\|_{L^{p'}(\Omega,\omega_1)} + \|\rho/\omega_2\|_{L^{p'}(\Omega,\omega_2)}\big)\|(g,f)\|_X.
\end{aligned} \tag{3.6}$$

Therefore, by (i), (ii), (iii), (iv) and (v), $\mathcal{T}(u) \in X^*$ for each $(u,\nabla u) \in \mathcal{K}_\psi$ and the mapping $\mathcal{T}$ is well defined.

Lemma 3.2. *The mapping $\mathcal{T}$ defined in* (3.1) *is monotone and coercive.*

Proof. (I) If $(u,\nabla u), (f,\nabla f) \in \mathcal{K}_\psi$, then

$$\begin{aligned}
&\mathcal{T}(u) - \mathcal{T}(f) \\
&= \left(\mathcal{H}(x,u,\nabla u)\frac{\nu_2}{\omega_1} + |u|^{p-2}u - \frac{\rho_0}{\omega_1}, \mathcal{A}(x,\nabla u) + \mathcal{B}(x,\nabla u)\frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2}\right) \\
&\quad - \left(\mathcal{H}(x,f,\nabla f)\frac{\nu_2}{\omega_1} + |f|^{p-2}f - \frac{\rho_0}{\omega_1}, \mathcal{A}(x,\nabla f) + \mathcal{B}(x,\nabla f)\frac{\nu_1}{\omega_2} - \frac{\rho}{\omega_2}\right) \\
&= \Bigg(\mathcal{H}(x,u,\nabla u)\frac{\nu_2}{\omega_1} - \mathcal{H}(x,f,\nabla f)\frac{\nu_2}{\omega_1} + |u|^{p-2}u - |f|^{p-2}f, \\
&\qquad \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla f) + \mathcal{B}(x,\nabla u)\frac{\nu_1}{\omega_2} - \mathcal{B}(x,\nabla f)\frac{\nu_1}{\omega_2}\Bigg).
\end{aligned} \tag{3.7}$$

Then by assumptions (H2), (H6) and (H10) we have

$$\begin{aligned}
&\langle \mathcal{T}(u) - \mathcal{T}(f), (u,\nabla u) - (f,\nabla f)\rangle \\
&\quad= \int_\Omega \big(\mathcal{H}(x,u,\nabla u) - \mathcal{H}(x,f,\nabla f)\big)(u-f)\, \nu_2\, dx \\
&\qquad+ \int_\Omega \big(|u|^{p-2}u - |f|^{p-2}f\big)(u-f)\, \omega_1\, dx \\
&\qquad+ \int_\Omega \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla f)\big)\cdot\nabla(u-f)\, \omega_2\, dx \\
&\qquad+ \int_\Omega \Big(\mathcal{B}(x,\nabla u) - \mathcal{B}(x,\nabla f)\Big)\cdot(\nabla u - \nabla f)\, \nu_1\, dx > 0,
\end{aligned}$$

since, by Proposition 2.3,

$$(|u|^{p-2}u - |f|^{p-2}f)(u-f) \geq \beta_p(|u|+|f|)^{p-2}|u-f| > 0$$

(if $u \neq f$). Hence, $\mathcal{T}$ is monotone.
(II) Let $(f,\nabla f) \in \mathcal{K}_\psi$ be fixed. For each $(u,\nabla u) \in \mathcal{K}_\psi$, by assumptions (H6) and (H10), we have

$$\begin{aligned}
&\langle T(u) - T(f), (u,\nabla u) - (f,\nabla f)\rangle \\
&\quad = \int_\Omega \big(\mathcal{H}(x,u,\nabla u) - \mathcal{H}(x,f,\nabla f)\big)(u-f)\,\nu_2\,dx \\
&\qquad + \int_\Omega \big(|u|^{p-2}u - |f|^{p-2}f\big)(u-f)\,\omega_1\,dx \\
&\qquad + \int_\Omega \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla f)\big)\cdot\nabla(u-f)\,\omega_2\,dx \\
&\qquad + \int_\Omega \Big(\mathcal{B}(x,\nabla u) - \mathcal{B}(x,\nabla f)\Big)\cdot(\nabla u - \nabla f)\,\nu_1\,dx \\
&\quad > \int_\Omega \big(|u|^{p-2}u - |f|^{p-2}f\big)(u-f)\,\omega_1\,dx \\
&\qquad + \int_\Omega \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla f)\big)\cdot\nabla(u-f)\,\omega_2\,dx.
\end{aligned} \tag{3.8}$$

By (H3) and (3.2), we obtain

$$\begin{aligned}
&\int_\Omega \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla f)\big)\cdot\nabla(u-f)\,\omega_2\,dx \\
&\quad = \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla u\,\omega_2\,dx + \int_\Omega \mathcal{A}(x,\nabla f)\cdot\nabla f\,\omega_2\,dx \\
&\qquad - \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla f\,\omega_2\,dx - \int_\Omega \mathcal{A}(x,\nabla f)\cdot\nabla u\,\omega_2\,dx \\
&\quad \geq \lambda_1 \int_\Omega |\nabla u|^p\omega_2\,dx + \lambda_1\int_\Omega |\nabla f|^p\omega_2\,dx \\
&\qquad - \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla f\,\omega_2\,dx - \int_\Omega \mathcal{A}(x,\nabla f)\cdot\nabla u\,\omega_2\,dx \\
&\quad \geq \lambda_1 \int_\Omega |\nabla u|^p\omega_2\,dx + \lambda_1\int_\Omega |\nabla f|^p\omega_2\,dx \\
&\qquad - \|h_1\|_{L^\infty(\Omega)}\|\nabla u\|^{p-1}_{L^p(\Omega,\omega_2)}\|\nabla f\|_{L^p(\Omega,\omega_2)} \\
&\qquad - \|h_1\|_{L^\infty(\Omega)}\|\nabla f\|^{p-1}_{L^p(\Omega,\omega_2)}\|\nabla u\|_{L^p(\Omega,\omega_2)}.
\end{aligned} \tag{3.9}$$

Moreover, by (3.5) we have

$$
\begin{aligned}
&\int_\Omega \big(|u|^{p-2}u - |f|^{p-2}f\big)(u-f)\,\omega_1\,dx \\
&\quad = \int_\Omega |u|^p\omega_1\,dx + \int_\Omega |f|^p\,\omega_1\,dx \\
&\qquad - \int_\Omega |u|^{p-2}u\,f\,\omega_1\,dx - \int_\Omega |f|^{p-2}f\,u\,\omega_1\,dx \\
&\quad \geq \int_\Omega |u|^p\omega_1\,dx + \int_\Omega |f|^p\omega_1\,dx \\
&\qquad - \|u\|^{p-1}_{L^p(\Omega,\omega_1)}\|f\|_{L^p(\Omega,\omega_1)} - \|f\|^{p-1}_{L^p(\Omega,\omega_1)}\|u\|_{L^p(\Omega,\omega_1)}.
\end{aligned}
\tag{3.10}
$$

Hence, by (3.8), (3.9) and (3.10) we obtain

$$
\begin{aligned}
&\langle \mathcal{T}(u) - \mathcal{T}(f), (u,\nabla u) - (f,\nabla f)\rangle \\
&\geq C_1\Big(\|u\|^p_{L^p(\Omega,\omega_1)} + \|\nabla u\|^p_{L^p(\Omega,\omega_2)} + \|f\|^p_{L^p(\Omega,\omega_1)} + \|\nabla f\|_{L^p(\Omega,\omega_2)}\Big) \\
&\quad - C_2\Big(\|\nabla u\|^{p-1}_{L^p(\Omega,\omega_2)}\|\nabla f\|_{L^p(\Omega,\omega_2)} + \|\nabla f\|^{p-1}_{L^p(\Omega,\omega_2)}\|\nabla u\|_{L^p(\Omega,\omega_2)} \\
&\quad + \|u\|^{p-1}_{L^p(\Omega,\omega_1)}\|f\|_{L^p(\Omega,\omega_1)} + \|f\|^{p-1}_{L^p(\Omega,\omega_1)}\|u\|_{L^p(\Omega,\omega_1)}\Big),
\end{aligned}
\tag{3.11}
$$

where $C_1 = \min\{1,\lambda_1\}$ and $C_2 = \max\{1, \|h_1\|_{L^\infty(\Omega)}\}$. To estimate the right-hand side of (3.11) from below, we use the inequality $\big(\sum_{j=1}^4 c_j\big)^q \leq 4^q \sum_{j=1}^4 c_j^q$ for all $c_j \geq 0$ $(j = 1,2,3,4)$ and $q > 0$. We have

(a) $$
\begin{aligned}
&(\|(u,\nabla u)\|_X + \|(f,\nabla f)\|_X)^p \\
&\quad = (\|u\|_{L^p(\Omega,\omega_1)} + \|\nabla u\|_{L^p(\Omega,\omega_2)} + \|f\|_{L^p(\Omega,\omega_1)} + \|\nabla f\|_{L^p(\Omega,\omega_2)})^p \\
&\quad \leq 4^p\Big(\|u\|^p_{L^p(\Omega,\omega_1)} + \|\nabla u\|^p_{L^p(\Omega,\omega_1)} + \|f\|^p_{L^p(\Omega,\omega_1)} + \|\nabla f\|^p_{L^p(\Omega,\omega_2)}\Big);
\end{aligned}
$$

(b) $$
\begin{aligned}
&\|\nabla u\|^{p-1}_{L^p(\Omega,\omega_2)}\|\nabla f\|_{L^p(\Omega,\omega_2)} + \|u\|^{p-1}_{L^p(\Omega,\omega_1)}\|f\|_{L^p(\Omega,\omega_1)} \\
&\quad \leq \|(u,\nabla u)\|^{p-1}_X\|(f,\nabla f)\|_X \\
&\quad \leq (\|(u,\nabla u)\|_X + \|(f,\nabla f)\|_X)^{p-1}\|(f,\nabla f)\|_X;
\end{aligned}
$$

(c) $$
\begin{aligned}
&\|\nabla u\|_{L^p(\Omega,\omega_2)}\|\nabla f\|^{p-1}_{L^p(\Omega,\omega_2)} + \|u\|_{L^p(\Omega,\omega_1)}\|f\|^{p-1}_{L^p(\Omega,\omega_1)} \\
&\quad \leq \|(u,\nabla u)\|_X\|(f,\nabla f)\|^{p-1}_X \\
&\quad \leq (\|(u,\nabla u)\|_X + \|(f,\nabla f)\|_X)\|(f,\nabla f)\|^{p-1}_X.
\end{aligned}
$$

Now using $(a), (b)$ and (c), from (3.11) we obtain

$$\begin{aligned}&\langle \mathcal{T}(u)-\mathcal{T}(f),(u,\nabla u)-(f,\nabla f)\rangle\\ &\quad\geq \frac{1}{4^p}\,C_1(\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X)^p\\ &\qquad -\,C_2(\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X)^{p-1}\|(f,\nabla f)\|_X\\ &\qquad -\,C_2(\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X)\,\|(f,\nabla f)\|_X^{p-1}.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\langle \mathcal{T}(u)-\mathcal{T}(f),(u,\nabla u)-(f,\nabla f)\rangle}{\|(u,\nabla u)-(f,\nabla f)\|_X}\\ &\quad\geq \frac{\langle \mathcal{T}(u)-\mathcal{T}(f),(u,\nabla u)-(f,\nabla f)\rangle}{\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X}\\ &\quad\geq \frac{1}{4^p}\,C_1(\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X)^{p-1}\\ &\qquad -C_2(\|(u,\nabla u)\|_X+\|(f,\nabla f)\|_X)^{p-2}\|(f,\nabla f)\|_X\\ &\qquad -\,C_2\|(f,\nabla f)\|_X^{p-1}.\end{aligned}\tag{3.12}$$

For each sequence $(u_k,\nabla u_k)\in\mathcal{K}_\psi$ with $\|(u_k,\nabla u_k)\|_X\to\infty$, we have

$$\|(u_k,\nabla u_k)\|_X+\|(f,\nabla f)\|_X\geq\|(u_k,\nabla u_k)\|_X\to\infty.$$

It follows that

$$\begin{aligned}&\frac{1}{4^p}\,C_1\big(\|(u_k,\nabla u_k)\|_X+\|(f,\nabla f)\|_X\big)^{p-1}\\ &\qquad -C_2\big(\|(u_k,\nabla u_k)\|_X+\|(f,\nabla f)\|_X\big)^{p-2}\|(f,\nabla f)\|_X\\ &\quad=\left(\frac{1}{4^p}\,C_1-C_2\frac{\|(f,\nabla f)\|_X}{\|(u_k,\nabla u_k)\|_X+\|(f,\nabla f)\|_X}\right)\\ &\qquad\times\big(\|(u_k,\nabla u_k)\|_X+\|(f,\nabla f)\|_X\big)^{p-1}\to\infty,\quad\text{as } k\to 0.\end{aligned}\tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\frac{\langle \mathcal{T}(u_k)-\mathcal{T}(f),(u_k,\nabla u_k)-(f,\nabla f)\rangle}{\|(u_k,\nabla u_k)-(f,\nabla f)\|_X}\to\infty,\quad\text{as}\quad k\to\infty.$$

Therefore, $\mathcal{T}$ is coercive in $\mathcal{K}_\psi$. □

Lemma 3.3. *The mapping $\mathcal{T}$ defined in* (3.1) *is weakly continuous.*

Proof. If $(u,\nabla u)$, $(f,\nabla f)\in\mathcal{K}_\psi$, then

$$\begin{aligned}&\langle \mathcal{T}(u)-\mathcal{T}(f),(f,\nabla f)\rangle\\ &=\int_\Omega\big(\mathcal{B}(x,\nabla u)-\mathcal{B}(x,\nabla f)\big)\cdot\nabla f\,\nu_1\,dx+\int_\Omega\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla f)\big)\cdot\nabla f\,\omega_2\,dx\\ &+\int_\Omega\big(\mathcal{H}(x,u,\nabla u)-\mathcal{H}(x,f,\nabla f)\big)\,f\,\nu_2\,dx+\int_\Omega\big(|u|^{p-2}u-|f|^{p-2}f\big)\,f\,\omega_1\,dx.\end{aligned}$$

The mapping $\mathcal{T}$ is weakly continuous on $\mathcal{K}_\psi$ if $\mathcal{T}(u_m)$ converges to $\mathcal{T}(u)$ weakly in X^*, i.e., $\langle \mathcal{T}(u_m),(f,\nabla f)\rangle \to \langle \mathcal{T}(u),(f,\nabla f)\rangle$ whenever $(u_m,\nabla u_m)$, $(u,\nabla u)\in\mathcal{K}_\psi$, $(u_m,\nabla u_m)\to(u,\nabla u)$ in X. It suffices to prove that $\langle \mathcal{T}(u_m)-\mathcal{T}(u),(f,\nabla f)\rangle \to 0$, for $(f,\nabla f)\in X$.
(a) We define the operators $G_j:\mathcal{K}_\psi\to L^{q'}(\Omega,\nu_1)$ (for $j=1,2,\dots,n$) by the formula

$$(G_j u)(x)=\mathcal{B}_j(x,\nabla u).$$

We now show that the operator G_j is bounded and continuous.
(i) By (H8) and Remark 2.2(i) we have

$$\begin{aligned}
\|G_j u\|^{q'}_{L^{q'}(\Omega\nu_1)} &= \int_\Omega |G_j u(x)|^{q'}\,\nu_1\,dx\\
&= \int_\Omega |\mathcal{B}_j(x,\nabla u)|^{q'}\,\nu_1\,dx\\
&\le \int_\Omega \left(h_2|\nabla u|^{q/q'}\right)^{q'}\,\nu_1\,dx\\
&\le \|h_2\|^{q'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{q}\,\nu_1\,dx\\
&\le C^q_{1,2}\|h_2\|^{q'}_{L^\infty(\Omega)}\left(\int_\Omega |\nabla u|^{p}\omega_2\,dx\right)^{q/p}\\
&= C^q_{1,2}\|h_2\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^p(\Omega,\omega_2)}\\
&\le C^q_{1,2}\|h_2\|^{q'}_{L^\infty(\Omega)}\|(u,\nabla u)\|^q_X.
\end{aligned}$$

Hence, G_j is bounded.
(ii) If $(u_m,\nabla u_m)\to(u,\nabla u)\in\mathcal{K}_\psi$, then $\nabla u_m\to\nabla u$ in $L^p(\Omega,\omega_2)$. By Theorem 2.1 there exists a subsequence $\{u_{m_k}\}$ and functions $\Phi_1\in L^p(\Omega,\omega_1)$, $\Phi_2\in L^p(\Omega,\omega_2)$ such that

$$\begin{aligned}
&u_{m_k}(x)\to u(x) \text{ a.e. in } \Omega;\\
&|u_{m_k}(x|\le \Phi_1(x) \text{ a.e. in } \Omega;\\
&D_j u_{m_k}\to D_j u(x) \text{ a.e. in } \Omega;\\
&|\nabla u_{m_k}(x)|\le \Phi_2(x) \text{ a.e. in } \Omega.
\end{aligned}\tag{3.14}$$

Next, applying (H8), we obtain

$$\begin{aligned}
|G_j u_{m_k}-G_j u|^{q'}\nu_1 &= |\mathcal{B}_j(x,\nabla u_{m_k})-\mathcal{B}_j(x,\nabla u)|^{q'}\nu_1\\
&\le C_q\big(|\mathcal{B}_j(x,\nabla u_{m_k})|^{q'}+|\mathcal{B}_j(x,\nabla u)|^{q'}\big)\,\nu_1\,dx\\
&\le C_q\big[(h_2|\nabla u_{m_k}|^{q-1})^{q'}+(h_2|\nabla u|^{q-1})^{q'}\big]\,\nu_1\\
&\le 2C_q\|h_2\|^{q'}_{L^\infty(\Omega)}\Phi_2^q\,\nu_1\in L^1(\Omega),
\end{aligned}$$

where C_q depends only on q and by Remark 2.2(i),

$$\int_\Omega \Phi_2^q \nu_1\,dx \le C_{1,2}^q \left(\int_\Omega \Phi_2^p \omega_2\,dx \right)^{q/p}.$$

By condition (H5), we have $G_j u_{m_k}(x) = \mathcal{B}_j(x,\nabla u_{m_k}) \to \mathcal{B}_j(x,\nabla u) = G_j u(x)$, as $m_k \to \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\nu_1)} \to 0,$$

that is, $Gu_{m_k} \to Gu$ in $L^{q'}(\Omega,\nu_1)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$G_j u_m \to G_j u \quad \text{in} \quad L^{q'}(\Omega,\nu_1). \tag{3.15}$$

(b) We define the operators $F_j : \mathcal{K}_\psi \to L^{p'}(\Omega,\omega_2)$ $(j = 1,2,\ldots,n)$ by the formula

$$(F_j u)(x) = \mathcal{A}_j(x,\nabla u).$$

We show that this operator is bounded and continuous.
(i) Using (H4), we obtain

$$\begin{aligned}
\|F_j u\|_{L^{p'}(\Omega\omega_2)}^{p'} &= \int_\Omega |F_j u(x)|^{p'}\,\omega_2\,dx \\
&= \int_\Omega |\mathcal{A}_j(x,\nabla u)|^{p'}\,\omega_2\,dx \\
&\le \int_\Omega \left(h_1 |\nabla u|^{p/p'} \right)^{p'} \omega_2\,dx \\
&\le \|h_1\|_{L^\infty(\Omega)}^{p'} \int_\Omega |\nabla u|^p\,\omega_2\,dx \\
&= \|h_1\|_{L^\infty(\Omega)}^{p'} \|\nabla u\|_{L^p(\Omega,\omega_2)}^p \\
&\le \|h_1\|_{L^\infty(\Omega)}^{p'} \|(u,\nabla u)\|_X^p.
\end{aligned}$$

Hence, F_j is bounded.
(ii) If $(u_m,\nabla u_m) \to (u,\nabla u) \in \mathcal{K}_\psi$, then $\nabla u_m \to \nabla u$ in $L^p(\Omega,\omega_2)$. By (3.14) and (H4) we obtain

$$\begin{aligned}
|F_j u_{m_k} - F_j u|^{p'}\,\omega_2 &= |\mathcal{A}_j(x,\nabla u_k) - \mathcal{A}_j(x,\nabla u)|^{p'}\,\omega_2 \\
&\le C_p \left(|\mathcal{A}_j(x,\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,\nabla u)|^{p'} \right) \omega_2 \\
&\le C_p \left(h_1^{p'} |\nabla u_{m_k}|^p + h_1^{p'} |\nabla u|^p \right) \omega_2 \\
&\le 2C_p \|h_1\|_{L^\infty(\Omega)}^{p'} \Phi_2^p \omega_2 \in L^1(\Omega),
\end{aligned}$$

where C_p depends only on p.

By condition (H1) we have $F_j u_{m_k}(x) = \mathcal{A}_j(x, \nabla u_{m_k}) \to \mathcal{A}_j(x, \nabla u) = F_j u(x)$, as $m_k \to \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_2)} \to 0,$$

that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega,\omega_2)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_2). \tag{3.16}$$

(c) We define the operator $H : \mathcal{K}_\psi \to L^{s'}(\Omega,\nu_2)$ by the formula

$$(Hu)(x) = \mathcal{H}(x,u,\nabla u).$$

(i) Using (H12) and Remark 2.2(i), we have

$$\begin{aligned}
\|Hu\|^{s'}_{L^{s'}(\Omega,\nu_2)} &= \int_\Omega |Hu|^{s'} \nu_2\,dx = \int_\Omega |\mathcal{H}(x,u,\nabla u)|^{s'} \nu_2\,dx \\
&\le \int_\Omega \left(h_3 |u|^{s/s'} + h_4 |\nabla u|^{s/s'} \right)^{s'} \nu_2\,dx \\
&\le C_s \int_\Omega \left(h_3^{s'} |u|^s + h_4^{s'} |\nabla u|^s \right) \nu_2\,dx \\
&\le C_s \left(\|h_3\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u|^s \nu_2\,dx + \|h_4\|_{L^\infty(\Omega)} \int_\Omega |\nabla u|^s \nu_2\,dx \right) \\
&\le C_s \left[\|h_3\|^{s'}_{L^\infty(\Omega)} C_{2,1}^s \left(\int_\Omega |u|^p \omega_1\,dx \right)^{s/p} \right. \\
&\qquad \left. + \|h_4\|^{s'}_{L^\infty(\Omega)} C_{2,2}^s \left(\int_\Omega |\nabla u|^p \omega_2\,dx \right)^{s/p} \right] \\
&\le C_s\, C \left[\left(\int_\Omega |u|^p \omega_1\,dx \right)^{s/p} + \left(\int_\Omega |\nabla u|^p \omega_2\,dx \right)^{s/p} \right] \\
&\le C_s\, C\, \|(u,\nabla u)\|_X^s,
\end{aligned}$$

where $C = \max\{C_{2,1}^s \|h_3\|^{s'}_{L^\infty(\Omega}, C_{2,2}^s \|h_4\|^{s'}_{L^\infty(\Omega)}\}$.

(ii) If $(u_m, \nabla u_m) \to (u, \nabla u) \in \mathcal{K}_\psi$, analogously to what was demonstrated with the operators G, F_j and by (3.14),

$$\begin{aligned}
|H(u_{m_k}) - H(u)|^{s'} \nu_2 &= |\mathcal{H}(x,u_{m_k},\nabla u_{m_k}) - \mathcal{H}(x,u,\nabla u)|^{s'} \nu_2 \\
&\le C_s \left(|\mathcal{H}(x,u_{m_k},\nabla u_{m_k})|^{s'} + |\mathcal{H}(x,u,\nabla u)|^{s'} \right) \nu_2
\end{aligned}$$

$$\leq C_s\left[\left(h_3|u_{m_k}|^{s/s'}+h_4|\nabla u_{m_k}|^{s/s'}\right)^{s'}+\left(h_3|u|^{s/s'}+h_4|\nabla u|^{s/s'}\right)^{s'}\right]\nu_2$$
$$\leq C_s\left[\left(\|h_3\|^{s'}_{L^\infty(\Omega)}|u_{m_k}|^s+\|h_4\|_{L^\infty(\Omega)}|\nabla u_{m_k}|^s\right)\right.$$
$$\left.+\left(\|h_3\|^{s'}_{L^\infty(\Omega)}|u|^s+\|h_4\|^{s'}_{L^\infty(\Omega)}|\nabla u|^s\right)\right]\nu_2$$
$$\leq 2C_s(\|h_3\|^{s'}_{L^\infty(\Omega)}\,\Phi_1^s+\|h_4\|^{s'}_{L^\infty(\Omega)}\,\Phi_2^s)\,\nu_2\in L^1(\Omega),$$

since, by Remark 2.2(ii),

$$\int_\Omega \Phi_1^s\nu_2\,dx\leq C_{2,1}^s\left(\int_\Omega \Phi_1^p\omega_1\,dx\right)^{s/p} \text{ and } \int_\Omega \Phi_2^s\nu_2\,dx\leq C_{2,2}^s\left(\int_\Omega \Phi_2^p\omega\,dx\right)^{s/p}.$$

By condition (H9) we have $Hu_{m_k}(x)=\mathcal{H}(x,u_{m_k},\nabla u_{m_k})\to\mathcal{H}(x,u,\nabla u)=Hu(x)$, as $m_k\to\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|Hu_{m_k}-Hu\|_{L^{s'}(\Omega,\nu_2)}\to 0,$$

that is, $Hu_{m_k}\to Hu$ in $L^{s'}(\Omega,\nu_2)$. From the Convergence Principle in Banach spaces (see Proposition 10.13 in [20]) we conclude that

$$Hu_m\to Hu \quad\text{in}\quad L^{s'}(\Omega,\nu_2). \tag{3.17}$$

(d) We define the operator $J:\mathcal{K}_\psi\to L^{p'}(\Omega,\omega_1)$ by the formula

$$(Ju)(x)=|u(x)|^{p-2}u(x).$$

(i) If $(u,\nabla u)\in\mathcal{K}_\psi$, then

$$\begin{aligned}\|Ju\|^{p'}_{L^{p'}(\Omega,\omega_1)}&=\int_\Omega |Ju(x)|^{p'}\omega_1\,dx\\&=\int_\Omega |u|^{(p-1)p'}\omega_1\,dx\\&=\int_\Omega |u|^p\,\omega_1\,dx\\&\leq\|(u,\nabla u)\|^p_X.\end{aligned}$$

(ii) If $(u_m,\nabla u_m)\to(u,\nabla u)\in\mathcal{K}_\psi$, analogously to what was demonstrated with the operators G, F_j, H and by (3.14), we obtain

$$Ju_m\to Ju \quad\text{in}\quad L^{p'}(\Omega,\omega_1). \tag{3.18}$$

Therefore, if $(u_m, \nabla u_m) \to (u, \nabla u)$ in $\mathcal{K}_\psi$, by Remark 2.2(i), (ii) we have

$$
\begin{aligned}
&|\langle \mathcal{T}(u_m) - \mathcal{T}(u), (f, \nabla f)\rangle| \\
&= \Bigg| \sum_{j=1}^n \int_\Omega \Big(\mathcal{B}_j(x, \nabla u_m) - \mathcal{B}_j(x, \nabla u) \Big) D_j f \, \nu_1 \, dx \\
&\quad + \sum_{j=1}^n \int_\Omega \Big(\mathcal{A}_j(x, \nabla u_m) - \mathcal{A}_j(x, \nabla u) \Big) D_j f \, \omega_2 \, dx \\
&\quad + \int_\Omega \Big(\mathcal{H}(x, u_m, \nabla u_m) - \mathcal{H}(x, u, \nabla u) \Big) f \, \nu_2 \, dx \\
&\quad + \int_\Omega (|u_m|^{p-2} u_m - |u|^{p-2} u) \, f \, \omega_1 \, dx \Bigg| \\
&\le \sum_{j=1}^n \int_\Omega |G_j u_m - G_j u| |D_j f| \, \nu_1 \, dx + \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u| \, |D_j f| \, \omega_2 \, dx \\
&\quad + \int_\Omega |H u_m - H u| |f| \, \nu_2 \, dx + \int_\Omega |J u_m - J u| |f| \omega_1 \, dx \\
&\le \sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega, \nu_1)} \|D_j f\|_{L^q(\Omega, \nu_1)} \\
&\quad + \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \|D_j f\|_{L^p(\Omega, \omega_2)} \\
&\quad + \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} \|f\|_{L^s(\Omega, \nu_2)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \|f\|_{L^p(\Omega, \omega_1)} \\
&\le C_{1,2} \Bigg(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega, \nu_1)} \Bigg) \|\nabla f\|_{L^p(\Omega, \omega_2)} \\
&\quad + \Bigg(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \Bigg) \|\nabla f\|_{L^p(\Omega, \omega_2)} \\
&\quad + C_{2,1} \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} \|f\|_{L^p(\Omega, \omega_1)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \|f\|_{L^p(\Omega, \omega_1)} \\
&\le \Bigg(C_{1.2} \sum_{j=1}^n \|G j u_m - G j u\|_{L^{q'}(\Omega, \nu_1)} + \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega, \omega_2)} \\
&\quad + C_{2,1} \|H u_m - H u\|_{L^{s'}(\Omega, \nu_2)} + \|J u_m - J u\|_{L^{p'}(\Omega, \omega_1)} \Bigg) \|(f, \nabla f)\|_X.
\end{aligned}
$$

Hence, using (3.15), (3.16), (3.17) and (3.18), we have

$$\langle \mathcal{T}(u_m) - \mathcal{T}(u), (f, \nabla f)\rangle \to 0$$

as $m \to \infty$, that is, $\mathcal{T}$ is weakly continuous. □

4. Proof of Theorem 1.1.

Existence of solution. Based on Lemma 3.1, Lemma 3.2 and Lemma 3.3, by Theorem 2.4 there exits an element $(u, \nabla u) \in \mathcal{K}_\psi$ such that

$$\langle T(u), (f, \nabla f) - (u, \nabla u)\rangle \geq 0$$

whenever $(f, \nabla f) \in \mathcal{K}_\psi$. For each $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, we have $u + \varphi - \psi = (u - \psi) + \varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Therefore, $(u + \varphi, \nabla u + \nabla\varphi) \in \mathcal{K}_\psi$ and $(u - \varphi, \nabla u - \nabla\varphi) \in \mathcal{K}_\psi$. Then

$$\langle \mathcal{T}(u), (\varphi, \nabla\varphi)\rangle = \langle \mathcal{T}(u), (u + \varphi, \nabla u + \nabla\varphi) - (u, \nabla u)\rangle \geq 0$$

and

$$\langle \mathcal{T}(u), (\varphi, \nabla\varphi)\rangle = -\langle \mathcal{T}(u), (u - \varphi, \nabla u - \nabla\varphi) - (u, \nabla u)\rangle \leq 0.$$

Therefore, $\langle \mathcal{T}(u), (\varphi, \nabla\varphi)\rangle = 0$, that is,

$$\begin{aligned}
&\int_\Omega \mathcal{B}(x, \nabla u)\cdot\nabla\varphi\, \nu_1\, dx + \int_\Omega \mathcal{A}(x, \nabla u)\cdot\nabla\varphi\, \omega_2\, dx \\
&\quad + \int_\Omega \mathcal{H}(x, u, \nabla u)\, \varphi\, \nu_2\, dx + \int_\Omega |u|^{p-2} u\, \varphi\, \omega_1\, dx \\
&= \int_\Omega \rho_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega \rho_j D_j\varphi\, dx,
\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, that is, u is a solution to problem (P).

Uniqueness. For the uniqueness, let u_1 and u_2 be two solutions to problem (P), with $u_i - \psi \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ $(i = 1, 2)$. Since $\psi, u_1, u_2 \in W^{1,p}(\Omega, \omega_1, \omega_2)$ and $u_1 - u_2 = (u_1 - \psi) - (u_2 - \psi) \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$, then

$$\begin{aligned}
&\int_\Omega \mathcal{B}(x, \nabla u_1)\cdot\nabla(u_1 - u_2)\, \nu_1\, dx + \int_\Omega \mathcal{A}(x, \nabla u_1)\cdot\nabla(u_1 - u_2)\, \omega_2\, dx \\
&\quad + \int_\Omega \mathcal{H}(x, u_1, \nabla u_1))(u_1 - u_2)\, \nu_2\, dx + \int_\Omega |u_1|^{p-2} u_1 (u_1 - u_2)\, \omega_1\, dx \\
&= \int_\Omega \rho_0 (u_1 - u_2)\, dx + \int_\Omega \rho\cdot\nabla(u_1 - u_2)\, dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \mathcal{B}(x, \nabla u_2)\cdot\nabla(u_1 - u_2)\, \nu_1\, dx + \int_\Omega \mathcal{A}(x, \nabla u_2)\cdot\nabla(u_1 - u_2)\, \omega_2\, dx \\
&\quad + \int_\Omega \mathcal{H}(x, u_2, \nabla u_2)(u_1 - u_2)\, \nu_2\, dx + \int_\Omega |u_2|^{p-2} u_2\, (u_1 - u_2)\, \omega_1\, dx \\
&= \int_\Omega \rho_0 (u_1 - u_2)\, dx + \int_\Omega \rho\cdot\nabla(u_1 - u_2)\, dx.
\end{aligned}$$

Hence,

$$\int_\Omega \Big(\mathcal{B}(x,\nabla u_1)-\mathcal{B}(x,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\,\nu_1\,dx$$
$$+\int_\Omega \Big(\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\,\omega_2\,dx$$
$$+\int_\Omega (\mathcal{H}(x,u_1,\nabla u_1)-\mathcal{H}(x,u_2,\nabla u_2))(u_1-u_2)\,\nu_2\,dx$$
$$+\int_\Omega (|u_1|^{p-2}u_1-|u_2|^{p-2}u_2)(u_1-u_2)\,\omega_1\,dx=0,$$

and by (H2), (H6), (H10) and Proposition 2.3 we have

$$\int_\Omega \Big(\mathcal{B}(x,\nabla u_1)-\mathcal{B}(x,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\,\nu_1\,dx=0$$
$$\int_\Omega \Big(\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\omega_2\,dx=0,$$
$$\int_\Omega \Big(\mathcal{H}(x,u_1,\nabla u_1)-\mathcal{H}(x,u_2,\nabla u_2)\Big)(u_1-u_2)\,\nu_2\,dx=0,$$
$$\int_\Omega (|u_1|^{p-2}u_1-|u_2|^{p-2}u_2)(u_1-u_2)\,\omega_1\,dx=0.$$

Thus,

$$\int_{\{\nabla u_1\neq\nabla u_2\}} \Big(\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2)\Big)\cdot\nabla(u_1-u_2)\omega_2\,dx\,dx=0.$$

By (H2) and $\omega_2\in A_p$, we obtain $|\{\nabla u_1\neq\nabla u_2\}|=0$ and $\nabla u_1=\nabla u_2$ a.e. in Ω. Since $u_1-u_2\in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we conclude that $u_1=u_2$ a.e. in Ω. Theorem 1.1 is thereby proved.

Example 4.1. Let $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. For $(x,y)\in\mathbb{R}^2$, $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, $\eta\in\mathbb{R}$, $p=4$ and $q=s=3$, consider the functions

$$\mathcal{A}((x,y),\xi)=\xi|\xi|^2\big(\cos(x/(x^2+y^2))+2\big)$$
$$\mathcal{B}((x,y),\xi)=\xi|\xi|\big(\sin(y/(x^2+y^2))+2\big),$$
$$\mathcal{H}((x,y),\eta,\xi)=\eta|\eta|(1+\cos^2(xy))$$
$$\omega_1(x,y)=(x^2+y^2)^{-1/2}\quad\text{and}\quad\omega_2(x,y)=(x^2+y^2)^{1/2}$$
$$\nu_1(x,y)=(x^2+y^2)^2\quad\text{and}\quad\nu_2(x,y)=(x^2+y^2)^{-1/8},$$
$$\rho_0(x,y)=\frac{\cos(xy)}{(x^2+y^2)}\quad\text{and}\quad\rho(x,y)=\left(\frac{\cos(xy)}{x^2+y^2},\frac{\sin(xy)}{x^2+y^2}\right),$$
$$\text{and}\quad\psi(x,y)=\sin(x+y).$$

Therefore, by Theorem 1.1, the problem (P) has a unique solution $u\in W^{1,4}(\Omega,\omega_1,\omega_2)$.

References

[1] Björn, A., Björn, J., Christensen, A., *Poincaré inequalities and A_p weights on bowties*, Preprint, 2022. arXiv:2202.07491v1.

[2] Bresch, D., Lemoine, J., Guíllen-Gonzalez, F., *A note on a degenerate elliptic equation with applications for lakes and seas*, Electron. J. Differential Equations **2004**(42) (2004), 1–13.

[3] Cavalheiro, A. C., *Existence results for Dirichlet problems with degenerate p-Laplacian*, Opuscula Math. **33**(3) (2013), 439–453.

[4] Cavalheiro, A. C., *Existence of solutions for Dirichlet problem of some degenerate quasilinear elliptic equations*, Complex Var. Elliptic Equ. **53**(2) (2008), 185–194.

[5] Cavalheiro, A. C., *Weighted Sobolev Spaces and Degenerate Elliptic Equations*, Cambridge Scholars Publishing, Newcastle upon Tyne, UK, 2023.

[6] Chipot, M., *Elliptic Equations: An Introductory Course*, Birkhäuser, Berlin, 2009.

[7] Colombo, M., *Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations: With Applications to the Vlasov-Poisson and Semigeostrophic Systems*, Publications on the Scuola Normale Superiore Pisa, 22, Pisa, 2017.

[8] Drábek, P., Kufner, A., Nicolosi, F., *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter, Berlin, 1997.

[9] Fabes, E., Kenig, C., Serapioni, R., *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116.

[10] Fučik, S., John, O., Kufner, A., *Function Spaces*, Noordhoff International Publ., Leyden, 1977.

[11] Garcia-Cuerva, J., Rubio de Francia, J. L., *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies 116, 1985.

[12] Heinonen, J., Kilpeläinen, T., Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, 1993.

[13] Kinderlehrer, D., Stampacchia, G., *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.

[14] Kufner, A., *Weighted Sobolev Spaces*, John Wiley & Sons, Germany, 1985.

[15] Kufner, A., Opic, B., *How to define reasonably weighted Sobolev spaces*, Comment. Math. Univ. Carolin. **25** (1984), 537–554.

[16] Kufner, A., Opic, B., *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics , Vol. 219, Longman Scientific & Technical, Harlow, 1990.

[17] Muckenhoupt, B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Am. Math. Soc. **165** (1972), 207–226.

[18] Torchinsky, A., *Real-Variable Methods in Harmonic Analysis*, Academic Press, San Diego, 1986.

[19] Turesson, B. O., *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lecture Notes in Math., vol. 1736, Springer-Verlag, 2000.

[20] Zeidler, E., *Nonlinear Functional Analysis and its Applications, vol. I*, Springer-Verlag, Berlin, 1990.

[21] Zeidler, E., *Nonlinear Functional Analysis and its Applications, vol. II/B*, Springer-Verlag, Berlin, 1990.

[22] Zhikov, V. V., *Weighted Sobolev spaces*, Sb. Math. **189**(8) (1998), 1139–1170.

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