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On a variant of Jessen–Mercer's inequality

Abstract. A new variant of Mercer's inequality [A.McD. Mercer, A variant of Jensen's inequality, *J. Inequal. Pure Appl. Math.* 4(4) (2003) Article 73] of Jessen's type is given. Moreover, versions of Chebyshev's inequality and Hardy–Littlewood– Pólya inequality for some abstract nonnegative linear functionals are obtained.

1. Introduction and motivation. Jensen's inequality [13] reads:

If f is a convex function on an interval I containing real numbers x_k $(k = 1, \dots, n)$, then

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \le \sum_{k=1}^{n} w_k f(x_k) \tag{1}$$

whenever w_k $(k = 1, \dots, n)$ are positive weights with the sum equal to 1.

The following integral version of (1) also holds true (see e.g. [27, Theorem 2.3]):

$$f\left(\int_E w(t)x(t)d\mu(t)\right) \le \int_E w(t)f(x(t))d\mu(t), \tag{2}$$

where $w : E \to [0, \infty]$ and $x : E \to I$ are functions that are integrable with respect to some nonnegative measure μ on a σ-algebra of subsets of a set E with $\int_E w(t)d\mu(t) = 1$.

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Mercer [19, Theorem 1.2] proved the following variant of (1).

Let α, β, x_k be real numbers with $\alpha \leq x_k \leq \beta$ $(k = 1, \dots, n)$. If f is a convex function defined on an interval containing α and β, then

$$f\left(\alpha + \beta - \sum_{k=1}^{n} w_k x_k\right) \leq f(\alpha) + f(\beta) - \sum_{k=1}^{n} w_k f(x_k). \tag{3}$$

The main step in his argumentation was [19, Lemma 1.3]:

if f is a convex function on a interval containing the numbers $\alpha \leq y \leq \beta$, then

$$f(\alpha + \beta - y) \leq f(\alpha) + f(\beta) - f(y).$$

Following Mercer's idea, by (2) and the above lemma we easy obtain the following observation.

Let $y = y(s)$ and $w = w(s)$ be μ-measurable functions, where $\alpha \leq y(s) \leq \beta$, $s \in E$ for certain real numbers $\alpha \leq \beta$, the function w is positive and integrable up to 1*.*

If f is a convex function on an interval containing α and β, then

$$f\left(\alpha + \beta - \int_E w(s)y(s)d\mu(s)\right) \leq f(\alpha) + f(\beta) - \int_E w(s)(f \circ y)(s)d\mu(s). \tag{4}$$

Mercer's inequality (3) gave rise to many generalizations and applications, see Abramovich et al. [1, 2, 3] Klaričić Bakula et al. [4, 5], Barić et al. [6], Cheung et al. [8], Gavrea [10], Matković et al. [16, 17, 18] and many others.

In [22] (cf. also [23]), Niezgoda gave the following generalization of (3):

$$f\left(\sum_{j=1}^{m} y_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} f(y_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(x_{ij}), \tag{5}$$

where $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, (y_j) is a real m-tuple, (x_{ij}) is a real $n \times m$ matrix and f is a convex function on an interval containing all of y_j and x_{ij}. He proved that (5) holds if the m-tuple (y_j) majorizes each row of the matrix (x_{ij}) [22, Theorem 2.1] or $\sum_{k=1}^{m}(y_k - x_{ik}) = 0$ and $(y_j) - (x_{ij})$ and (x_{ij}) are pairs of separable m-tuples for every $i = 1, 2, \dots, n$ [22, Theorem 3.1]. The key tool in both cases was Hardy–Littlewood–Pólya inequality (HLP-inequality, for short), see [11, 9, 20, 21]:

$$\sum_{j=1}^{m} p_j f(x_j) \leq \sum_{j=1}^{m} p_j f(y_j), \tag{6}$$

where f is a real convex function, (x_j), (y_j) are real m-tuples and (p_j) is an m-tuple with positive entries.

The integral version of inequality (6):

$$\int_E (f \circ x)(s)p(s)d\mu \leq \int_E (f \circ y)(s)p(s)d\mu, \tag{7}$$

where $x(t)$, $y(s)$, $p = p(t)$ are measurable and $p(x)$ is nonnegative with $\int p(t)d\mu > 0$, also holds true under appropriate assumptions about the integrand functions, see e.g. [7, 11, 26]. Continuous generalizations of inequality (5) were studied in [12, 15, 26].

The functionals $A(x) = \sum_{k=1}^{n} w_k x_k$ and $A(x) = \int_E w(t)x(t)d\mu(t)$, occurring in Jensen's inequalities (1) and (2), are fundamental examples of *linear means*.

An abstract linear mean is a linear nonnegative functional A acting on a certain real linear space L of some real-valued functions defined on a given nonempty set E such that L contains the function $\mathbb{I}$ constantly equal to 1 and $A(\mathbb{I}) = 1$.

For $x \in L$ and A being a linear mean on L, the following inequality established by Jessen [14] (see also [27, Theorem 2.4]):

$$f(A(x)) \leq A(f \circ x) \tag{8}$$

generalizes both versions of Jensen's inequalities (1) and (2).

An inequality generalizing in the same spirit two Mercer's variants of Jensen inequality (3) and (4), i.e.,

$$f(\alpha + \beta - A(z)) \leq f(\alpha) + f(\beta) - A(f \circ z), \tag{9}$$

where $z \in L$ and $\alpha \leq z(s) \leq \beta$, $s \in E$, was obtained in [8].

In Section 3 of this article, we will prove a counterpart of the inequality (5) for linear means and linear nonnegative functionals acting on abstract spaces of real bounded functions. Simultaneously, this result generalizes an integral version of the inequality given recently by the author in [26, Theorem 3] and provides an answer for open problem raised in [15]. The path to the goal leads through the generalizations of Chebyshev's inequality and HLP inequality for nonnegative functionals. This topic is considered in Section 2.

2. Versions of Chebyshev's inequality and HLP-inequality for nonnegative functionals. Let L be a class of some bounded real-valued functions measurable with respect to a given σ-algebra of subsets of a nonempty base set E, having the properties:

L1: $x, y \in L \Rightarrow \alpha x + \beta y \in L$ for all $\alpha, \beta \in \mathbb{R}$;

L2: the index functions I_Z of measurable sets $Z \subset E$ belong to L;

L3: $x, y \in L \Rightarrow xy \in L$.

Let B be a linear nonnegative functional on L, i.e. any real linear functional with $x \geq 0 \Rightarrow B(x) \geq 0$, $x \in L$. For linear functionals, nonnegativity is equivalent to isotonicity: $x \leq y \Rightarrow B(x) \leq B(y)$, $x, y \in L$.

The inequality (see e.g. [27, Section 7.1])

$$\int_E a(s)p(s)d\mu \int_E b(s)p(s)d\mu \leq \int_E a(s)b(s)p(s)d\mu \int_E p(s)d\mu, \tag{10}$$

where a, b and positive p are measurable functions with respect to some nonnegative measure μ on a σ-algebra of subsets of a set E holds for similarly ordered a and b, i.e.,

$$[a(s) - a(t)][b(s) - b(t)] \geq 0, \text{ for } s, t \in [0, 1],$$

provided that the integrals exist and $\int_E p(s)d\mu > 0$.

This is a newer version of the well-known classical inequality established by Chebyshev in years 1882–1883. There are several results which show that (10) is valid under more general conditions, see e.g. [24]. Note that $\int_E a(s)p(s)d\mu$ is an example of nonnegative functional B that acts on $a \in L$.

Here, we are particularly interested in a general variant of Chebyshev's inequality for nonnegative functionals.

Proposition 1. *Let $a, b \in L$ and B be a nonnegative linear functional on L. If a and b are similarly ordered, then*

$$B(a)B(b) \leq B(ab)B(\mathbb{I}). \tag{11}$$

Proof. Inequality (11) is a consequence of the following identity:

$$B(ab)B(\mathbb{I}) - B(a)B(b) = \frac{1}{2}B_s(B_t[a(s) - a(t)][b(s) - b(t)]),$$

where B_t and B_s are copies of B acting on functions of t variable or s variable, respectively. □

For further considerations we recall some well-known results on convex functions (see e.g. [28, Theorem 2.1.5]). Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. The function f is continuous on $\mathbb{R}$. There exist finite $f'_-(t)$ and $f'_+(t)$, i.e. the left-hand and the right-hand derivative of first order, respectively, for all $t \in \mathbb{R}$. If $t_1 < t_2$, then $f'_-(t_1) \leq f'_+(t_1) \leq f'_-(t_2) \leq f'_+(t_2)$. The subdifferential of f denoted by ∂f is the set of all functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $f'_-(t) \leq \phi(t) \leq f'_+(t)$. Every $\phi \in \partial f$ is a nondecreasing function. If f is nondecreasing convex, then f'_- and f'_+ are both nonnegative showing that ϕ is nonnegative, too. Moreover, for every $s, t \in \mathbb{R}$ and $\phi \in \partial f$,

$$f(s) - f(t) \geq (s - t)\phi(t). \tag{12}$$

It is known (see [7, 9, 21, 25]), that Chebyshev's type inequalities yields generalizations of HLP-inequalities. This is confirmed by the next result.

Proposition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function with $\phi \in \partial f$ and B be a linear nonnegative functional on L with $A(\mathbb{I}) > 0$.*

For $x, y \in L$ such that $f \circ x$, $f \circ y$, $\phi \circ x \in L$, the inequality

$$B(f \circ x) \leq B(f \circ y) \tag{13}$$

is valid if $y - x$ and x are similarly ordered and

$$B(y) = B(x) \quad \text{or} \quad \begin{cases} f \text{ is nondecreasing,} \\ B(y) \geq B(x). \end{cases} \tag{14}$$

Proof. By (12), $(f\circ y)(t)-(f\circ x)(t)\geq[y(t)-x(t)](\phi\circ x)(t)$ for all $t\in E$. The functions $f\circ y$, $f\circ x$, $(y-x)\cdot\phi\circ x$ are bounded, so belong to L. Isotonicity of B gives

$$B(f\circ y)-A(f\circ x)\geq B((y-x)\phi\circ x). \tag{15}$$

Clearly, x and $\phi\circ x$ are similarly ordered for any nondecreasing $\phi:\mathbb{R}\to\mathbb{R}$. By the hypothesis, $y-x$ and x are similarly ordered. As a consequence,

$$[(y(s)-x(s))-(y(t)-x(t))][x(s)-x(t)]^2[\phi\circ x(s)-\phi\circ x(t)]\geq 0$$

and $x(s)-x(t)=0$ implies $\phi\circ x(s)-\phi\circ x(t)=0$ for all $s,t\in E$. It follows that $y-x$ and $\phi\circ x$ are similarly ordered. Moreover, $\phi\circ x$ is bounded whenever y is bounded, because $\phi\in\partial f$ is nondecreasing and finite at every point of its domain $\mathbb{R}$.

Utilizing Chebyshev's type inequality (11), we get

$$B((y-x)\phi\circ x)\geq\frac{B(y-x)B(\phi\circ x)}{B(\mathbb{I})}. \tag{16}$$

Combining inequalities (15) and (16), we conclude (13) whenever (14), because every $\phi\in\partial f$ is nonnegative for nondecreasing f and consequently, $B(\phi\circ x)\geq 0$. □

Recently, related results in case of integrals were obtained by Barnett et al. [7, Theorems 6–7] and Otachel [26, Theorem 2].

3. A variant of Mercer's inequality in Jessen's sense. Let L and K be real linear spaces of some bounded real-valued measurable functions defined on given nonempty base sets E and F, respectively, that fulfil conditions L1–L3 from Section 2.

Consider a linear and nonnegative functional B on K and fix $0\leq b\in K$ such that $B(b)>0$. It is clear that the functional $h\mapsto\frac{B(hb)}{B(b)}$ is a linear mean defined for $h\in K$.

Theorem 1. *Let A be a linear mean on L, B be a linear nonnegative functional on K and $f:\mathbb{R}\to\mathbb{R}$ be a convex function with $f(L)\subset L$.*

Assume that $x_s=x_s(t)=x(s,t)$, $y=y(t)$, $s\in E$, $t\in F$, are bounded real-valued functions, $a=a(t)$, $b=b(t)$, $t\in F$ are bounded nonnegative functions with $a(t)+b(t)=1$, $t\in F$ and $B(b)>0$, such that x_s,y,a,b, $f\circ y,f\circ x_s,\phi\circ x_s\in K$, $s\in E$, $\phi\in\partial f$ and $B(ax_s),B(af\circ x_s)\in L$.

If $y-x_s$ and x_s are similarly ordered for every $s\in E$ and

$$B(y)=B(x_s),\ \ s\in E, \tag{17}$$

then

$$f\left(\frac{B(y)}{B(b)}-\frac{AB(ax_s)}{B(b)}\right)\leq\frac{B(f\circ y)}{B(b)}-\frac{AB(af\circ x_s)}{B(b)}. \tag{18}$$

Proof. By identity (17), for any $s \in E$ we have

$$f\left(\frac{B(y)}{B(b)} - \frac{B(ax_s)}{B(b)}\right) = f\left(\frac{B(bx_s)}{B(b)}\right). \tag{19}$$

Since $\frac{B(bx_s)}{B(b)}$ is a linear mean acting on x_s, applying Jessen's inequality (8), we get

$$f\left(\frac{B(bx_s)}{B(b)}\right) \le \frac{B(bf \circ x_s)}{B(b)}. \tag{20}$$

According to Proposition 2, inequality (13) leads to

$$\frac{B(af \circ x_s)}{B(b)} + \frac{B(bf \circ x_s)}{B(b)} = \frac{B(f \circ x_s)}{B(b)} \le \frac{B(f \circ y)}{B(b)}. \tag{21}$$

Thus, by (19), (20) and (21), we conclude that

$$f\left(\frac{B(y)}{B(b)} - \frac{B(ax_s)}{B(b)}\right) \le \frac{B(f \circ y)}{B(b)} - \frac{B(af \circ x_s)}{B(b)}, \quad s \in E. \tag{22}$$

According to our assumptions, both sides of inequality (22) are functions (of variable s) from the space L. Since A is a linear mean, we obtain

$$Af\left(\frac{B(y)}{B(b)} - \frac{B(ax_s)}{B(b)}\right) \le \frac{B(f \circ y)}{B(b)} - \frac{AB(af \circ x_s)}{B(b)}.$$

On the other hand, applying Jessen's inequality (8) second time, we see that

$$f\left(\frac{B(y)}{B(b)} - \frac{AB(ax_s)}{B(b)}\right) \le Af\left(\frac{B(y)}{B(b)} - \frac{B(ax_s)}{B(b)}\right).$$

Combining the above two inequalities, we derive inequality (18). □

Remark 1. If $a(t) \equiv 0$, then $b \equiv 1$ and consequently, inequalities (18) and (22) reduce to

$$f\left(\frac{B(y)}{B(\mathbb{I})}\right) \le \frac{B(f \circ y)}{B(\mathbb{I})},$$

which is a variant of Jessen's inequality (8) for the linear mean $\frac{B(y)}{B(\mathbb{I})}$.

In the example below we show that inequality (9) is a particular case of inequality (18) obtained in Theorem 1.

Example 1. For $B(y) = y_1 + y_2$, $y = (y_1, y_2) \in K = \mathbb{R}^2$ and an arbitrary chosen function $z = z(s) \in L$, where $\alpha \le z(s) \le \beta$, $s \in E$, for some $\alpha, \beta \in \mathbb{R}$, let us define

$$a = (1,0), \quad E_1 = \left\{ s \in E : \alpha \le z(s) < \frac{\alpha+\beta}{2} \right\},$$
$$b = (0,1), \quad E_2 = \left\{ s \in E : \frac{\alpha+\beta}{2} \le z(s) \le \beta \right\},$$

$$y = (\alpha, \beta), \quad x = \begin{cases} (z(s), \alpha + \beta - z(s)), & s \in E_1, \\ (0,0), & s \in E_2, \end{cases}$$

$$\tilde{y} = (\beta, \alpha), \quad \tilde{x} = \begin{cases} (z(s), \alpha + \beta - z(s)), & s \in E_2, \\ (0,0), & s \in E_1. \end{cases}$$

Additionally, assume the mathematical objects introduced above satisfy the remaining compliance conditions specified in Theorem 1. It is easily seen that $y - x_s$ and x_s are similarly ordered and $\tilde{y} - \tilde{x}_s$ and $\tilde{x}_s$ are similarly ordered, for $s \in E$ and for any function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned} B(b) &= 1, \\ B(y) = B(x_s) &= \alpha + \beta, \\ B(f \circ y) = B(f \circ \tilde{y}) &= f(\alpha) + f(\beta), \\ B(\tilde{y}) = B(\tilde{x}_s) &= \alpha + \beta, \\ B(af \circ x_s) &= (f \circ z)(s), \ s \in E_1, \\ B(ax_s) &= z(s), \ s \in E_1, \\ B(af \circ \tilde{x}_s) &= (f \circ z)(s), \ s \in E_2, \\ B(a\tilde{x}_s) &= z(s), \ s \in E_2. \end{aligned} \tag{23}$$

Based on Theorem 1, if f is convex and A is a linear mean on L such that $A(I_{E_k}) > 0, \ k = 1, 2$, we state that inequality (18) is valid for systems of objects A_1, B, y, x, a, b and $A_2, B, \tilde{y}, \tilde{x}, a, b$, where $A_k h = \frac{A(hI_{E_k})}{A(I_{E_k})}, \ h \in L, \ k = 1, 2$ are linear means on L. In fact, by (23) we obtain

$$f(\alpha + \beta - A_k(z)) \leq f(\alpha) + f(\beta) - A_k(f \circ z), \ k = 1, 2. \tag{24}$$

Multiplying inequalities (24) by $\varepsilon_k := A(I_{E_k}) > 0, \ k = 1, 2$ and summing over k, we get

$$\begin{aligned} &\varepsilon_1 f(\alpha + \beta - A_1(z)) + \varepsilon_2 f(\alpha + \beta - A_2(z)) \\ &\quad \leq f(\alpha) + f(\beta) - (\varepsilon_1 A_1 + \varepsilon_2 A_2)(f \circ z) = f(\alpha) + f(\beta) - A(f \circ z), \end{aligned} \tag{25}$$

because $\varepsilon_1 + \varepsilon_2 = 1$ and $\varepsilon_1 A_1 + \varepsilon_2 A_2 = A$.

On the other hand, the convexity of f ensures

$$\begin{aligned} &\varepsilon_1 f(\alpha + \beta - A_1(z)) + \varepsilon_2 f(\alpha + \beta - A_2(z)) \\ &\quad \geq f(\alpha + \beta - (\varepsilon_1 A_1 + \varepsilon_2 A_2)(z)) = f(\alpha + \beta - A(z)). \end{aligned} \tag{26}$$

Combining (25) and (26), we obtain (18).

The specification described in the next example reduces inequality (18) to Niezgoda's inequality (5) (cf. [22, Theorem 3.1, Corollary 3.3]).

Example 2. Using notation as in Theorem 1, let $L = \mathbb{R}^n$ and $K = \mathbb{R}^m$. For $y = (y_1, \ldots, y_m) \in K$, $z = (z_1, \ldots, z_n) \in L$ and given positive vectors $(p_1, \ldots, p_m)$ and $(w_1, \ldots, w_n)$ with $\sum_{s=1}^n w_s = 1$ define $A(z) = \sum_{s=1}^n w_s z_s$, the linear mean on L and $B(y) = \sum_{t=1}^{m} p_t y_t$, the nonnegative linear functional on K. Let $x = (x_{st})$ be a real matrix $n \times m$, $a = (1, \dots, 1, 0)$, $b = (0, \dots, 0, 1) \in \mathbb{R}^m$.

If $(y_1 - x_{s1}, \dots, y_m - x_{sm})$ and $(x_{s1}, \dots, x_{sm})$ are similarly ordered and the condition (17) holds in the form $\sum_{t=1}^{m} p_t y_t = \sum_{t=1}^{m} p_t x_{st}$, for every $s = 1, \dots, n$, then by Theorem 1 we get (18) in the form

$$f\left(\frac{1}{p_m}\left\{\sum_{t=1}^{m} p_t y_t - \sum_{t=1}^{m-1} p_t \sum_{s=1}^{n} w_s x_{st}\right\}\right) \le \frac{1}{p_m}\left\{\sum_{t=1}^{m} p_t f(y_t) - \sum_{t=1}^{m-1} p_t \sum_{s=1}^{n} w_s f(x_{st})\right\},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex function. If $p_t = p_m$ for every $t = 1, \dots, m$, the above inequality becomes (5).

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