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A new q -Laplace transform with many examples

ABSTRACT. In the spirit of Hahn 1949, the purpose of this paper is to introduce a new q -Laplace transform for a Jackson q -integral $\int_0^a f(t, q) d_q(t)$, with upper integration boundary $\frac{1}{s(1-q)}$. For this purpose we redefine this q -integral with a σ -algebra and a discrete measure supported at the points $x = aq^n$, $n \in \mathbb{N}$. Then we prove q -analogues of many well-known Laplace transform formulas, including the formula for the transform of the delta distribution. The paper concludes with a list of q -Laplace transforms for (multiple) q -hypergeometric series, some with function arguments in the first q -real numbers $\mathbb{R}_{\oplus q}$. Elsewhere, other q -real numbers are defined in similar style as function arguments in formal power series.

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1. Introduction

In q -difference equations, the derivatives of differential equations are replaced by q -derivatives. The theory of q -difference equations is not fully explored, and there is, in particular, a need to define a correct q -Laplace transform. Obviously, when this is known, all q -difference equations, which are q -analogues of the corresponding ordinary, homogeneous differential equations with constant coefficients, can be solved. The reason is that these differential equations have exponential and/or trigonometric solutions and our q -analogues of these two functions (and of the derivative as well) have the same q -Laplace transform. As a guide to the reader, who is assumed to have some knowledge about Laplace transforms, all the details will be explained systematically. Hint: All beginner's books on the Laplace transform give a series of formulas, which are often repeated in each book. The weak point in q -calculus is that there is no q -analogue of generalized integrals, although these are stated e.g. in the book by Gasper and Rahman [13]. This means that all formulas with generalized integrals for Laplace transforms, which change order of integration or which make a linear substitution in integrals, have no q -analogue. The experienced reader may now already guess which formulas for Laplace transforms that we cannot q -deform. We will, however, q -deform all the other, more transparent, ones.

The paper is organized as follows: In Section 1 we make a brief introduction to the subject. Section 2 introduces the notations, some of which can be found in our book [8]. In Section 3 we briefly repeat the first q -real number $\mathbb{R}_{\oplus q}$ from [10]. Section 4 presents a corrected version of the q -Laplace transform by Chung, Kim and Kwon [2], and in Subsection 4.1 the fundamental prerequisites for the Jackson q -integral are summarized. In Subsection 4.2, finally, we give a correct version of the q -Laplace transform. In Subsection 4.3 we only present some typical proofs of several q -hypergeometric q -Laplace transforms.

2. q -Calculus definitions

We now repeat some notations from [8]. Throughout, \equiv denotes a definition and \cong denotes a formal equality.

Definition 1. Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \text{Im}(\log q) \leq \pi + \delta$. This defines a simply connected domain in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)}.$$

The following notation is often used when long exponents appear.

$$\text{QE}(x) \equiv q^x.$$

Definition 2 ([8, p. 19]). The q -analogues of a complex number a , a natural number n and the factorial are defined as follows:

$$\{a\}_q \equiv \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} \setminus \{0, 1\},$$

$$\{n\}_q \equiv \sum_{k=1}^n q^{k-1}, \quad \{0\}_q = 0, \quad q \in \mathbb{C} \setminus \{0, 1\},$$

$$\{n\}_q! \equiv \prod_{k=1}^n \{k\}_q, \quad \{0\}_q! \equiv 1, \quad q \in \mathbb{C} \setminus \{0, 1\}.$$

Definition 3. The q -shifted factorial [8] is defined by

$$\langle a; q \rangle_n \equiv \prod_{m=0}^{n-1} (1 - q^{a+m}).$$

Sometimes we also use

$$(a; q)_n \equiv \prod_{m=0}^{n-1} (1 - aq^m).$$

Definition 4. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. The operator

$$\widetilde{\cdot} : \frac{\mathbb{C}}{\mathbb{Z}} \rightarrow \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$(1) \quad a \mapsto a + \frac{\pi i}{\log q}.$$

By (1) it follows that

$$\widetilde{\langle a; q \rangle_n} = \prod_{m=0}^{n-1} (1 + q^{a+m}),$$

where this time the tilde denotes the involution which changes a minus sign to a plus sign in all the n factors of $\langle a; q \rangle_n$.

For relatively prime m, l , the generalized tilde operator

$$\frac{\widetilde{\cdot}}{l} : \frac{\mathbb{C}}{\mathbb{Z}} \rightarrow \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$(2) \quad a \mapsto a + \frac{2\pi im}{l \log q}.$$

We also need another generalization of the tilde operator.

$$(3) \quad {}_k \widetilde{\langle a; q \rangle}_n \equiv \prod_{m=0}^{n-1} \left(\sum_{i=0}^{k-1} q^{i(a+m)} \right).$$

Formula (3) is used in (4).

The following simple congruence rules [8] follow from (2).

Theorem 1.

$$\begin{aligned} \widetilde{\frac{m}{l} a} \pm b &\equiv \frac{m}{l} \widetilde{(a \pm b)} \pmod{\frac{2\pi i}{\log q}}, \\ \sum_{k=1}^n \frac{1}{n} \widetilde{\pm a_k} &\equiv \sum_{k=1}^n \pm a_k \pmod{\frac{2\pi i}{\log q}}, \\ \frac{m}{l} \times \widetilde{a} &\equiv \frac{\widetilde{am}}{l} \pmod{\frac{2\pi i}{\log q}}, \end{aligned}$$

$$\text{QE}(\widetilde{\frac{m}{l} a}) = \text{QE}(a) e^{\frac{2\pi im}{l}},$$

where the second equation is a consequence of the fact that we work mod $\frac{2\pi i}{\log q}$.

Definition 5.

$$(4) \quad \langle \lambda; q \rangle_{kn} \equiv \langle \Delta(q; k; \lambda); q \rangle_n \equiv \prod_{m=0}^{k-1} \left\langle \frac{\lambda + m}{k}; q \right\rangle_n \times_k \left\langle \frac{\widetilde{\lambda + m}}{k}; q \right\rangle_n.$$

We also use the notation $\Delta(q; k; \lambda)$ as a parameter in q -hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector components. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q -factorials.

The last factor in (4) corresponds to k^{nk} .

Definition 6 ([8, (1.45)]). The Γ_q function is defined by

$$\Gamma_q(z) \equiv \begin{cases} \frac{\langle 1; q \rangle_\infty}{\langle z; q \rangle_\infty} (1-q)^{1-z}, & \text{if } 0 < |q| < 1; \\ \frac{\langle 1; q^{-1} \rangle_\infty}{\langle z; q^{-1} \rangle_\infty} (q-1)^{1-z} q^{\binom{z}{2}}, & \text{if } |q| > 1. \end{cases}$$

Definition 7 ([8, (1.49)]). Let S_r denote the additional poles of Γ_q , vertical if q is real and slanting if q is complex. Then the generalized Γ_q function, a function $(\mathbb{C} \setminus (\{Z \leq 0\} \cup S_r))^{p+r} \times \mathbb{C} \mapsto \mathbb{C}$, is defined as follows:

$$\Gamma_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_r \end{matrix} \right] \equiv \frac{\Gamma_q(a_1) \dots \Gamma_q(a_p)}{\Gamma_q(b_1) \dots \Gamma_q(b_r)}.$$

This is a modest attempt to present a new notation for q -calculus and in particular for q -hypergeometric series, which is compatible with the old notation. With this notation, q -hypergeometric function- and hypergeometric function equations become very similar.

Definition 8. Generalizing Heine series, we shall define a q -hypergeometric series by

$$\begin{aligned} & {}_{p+p'}\phi_{r+r'} \left[\begin{matrix} \hat{a}_1, \dots, \hat{a}_p \\ \hat{b}_1, \dots, \hat{b}_r \end{matrix} \middle| q; z \middle| \left| \frac{\prod_i f_i(k)}{\prod_j g_j(k)} \right. \right] \\ & \equiv \sum_{k=0}^{\infty} \frac{\langle \hat{a}_1; q \rangle_k \dots \langle \hat{a}_p; q \rangle_k}{\langle 1, \hat{b}_1; q \rangle_k \dots \langle \hat{b}_r; q \rangle_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+r+r'-p-p'} z^k \frac{\prod_i f_i(k)}{\prod_j g_j(k)}, \end{aligned}$$

where

$$\hat{a} \equiv a \vee \tilde{a} \vee \widetilde{\frac{m}{l}} a \vee_k \tilde{a} \vee \Delta(q; l; \lambda).$$

In case of $\Delta(q; l; \lambda)$ the index is adjusted accordingly. It is assumed that the denominator contains no zero factors, i.e. $\hat{b}_k \neq -l + \frac{2m\pi i}{\log q}$, $k = 1, \dots, r$, $l, m \in \mathbb{N}$ [18]. We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle \widehat{a}(k); q \rangle_k$ or $\langle s(k); q \rangle_k$ respectively.

The following definition, as in the one-variable case, allows easy limits for parameters to $\pm\infty$.

The first definition is a q -analogue of [19, (24), p. 38], in the spirit of Srivastava. The second definition is a q -analogue of [19, (24), p. 38] with the restraint [19, (29), p. 38], due to Karlsson. It will be clear from the context which of the definitions we use.

Definition 9 ([8, p. 367 f]). Let the vectors

$$(a), (b), (g_i), (h_i), (a'), (b'), (g'_i), (h'_i)$$

have lengths

$$A, B, G_i, H_i, A', B', G'_i, H'_i.$$

Let

$$1 + B + B' + H_i + H'_i - A - A' - G_i - G'_i \geq 0, \quad i = 1, \dots, n.$$

Then the generalized q -Kampé de Fériet function is defined by

$$\begin{aligned} & \Phi_{B+B':H_1+H'_1;\dots;H_n+H'_n}^{A+A':G_1+G'_1;\dots;G_n+G'_n} \left[\begin{array}{c} (\hat{a}) : (\hat{g}_1); \dots; (\hat{g}_n) \\ (\hat{b}) : (\hat{h}_1); \dots; (\hat{h}_n) \end{array} \middle| \begin{array}{c} \vec{q}; \vec{x} \\ \end{array} \middle| \begin{array}{c} (a') : (g'_1); \dots; (g'_n) \\ (b') : (h'_1); \dots; (h'_n) \end{array} \right] \\ & \equiv \sum_{\vec{m}} \frac{\langle (\hat{a}); q_0 \rangle_m (a')(q_0, m) \prod_{j=1}^n (\langle (\hat{g}_j); q_j \rangle_{m_j} ((g'_j)(q_j, m_j) x_j^{m_j}))}{\langle (\hat{b}); q_0 \rangle_m (b')(q_0, m) \prod_{j=1}^n (\langle (\hat{h}_j); q_j \rangle_{m_j} (h'_j)(q_j, m_j) \langle 1; q_j \rangle_{m_j})} \\ & \quad \times (-1)^{\sum_{j=1}^n m_j (1+H_j+H'_j-G_j-G'_j+B+B'-A-A')} \\ & \quad \times \text{QE} \left((B+B'-A-A') \binom{m}{2}, q_0 \right) \\ & \quad \times \prod_{j=1}^n \text{QE} \left((1+H_j+H'_j-G_j-G'_j) \binom{m_j}{2}, q_j \right). \end{aligned}$$

It is assumed that there are no zero factors in the denominator and that $(a')(q_0, m)$, $(g'_j)(q_j, m_j)$, $(b')(q_0, m)$, $(h'_j)(q_j, m_j)$ contain factors of the form $\langle a(\hat{k}); q \rangle_k$, $\langle s; q \rangle_k$, $\langle s(k); q \rangle_k$ or $\text{QE}(f(\vec{m}))$.

Definition 10 ([8, p. 368 f]). Let the vectors

$$(a), (b), (g_i), (h_i), (a'), (b'), (g'_i), (h'_i)$$

have lengths

$$A, B, G, H, A', B', G', H'.$$

Let

$$1 + B + B' + H + H' - A - A' - G - G' \geq 0.$$

Then the generalized q -Kampé de Fériet function is defined by

$$\begin{aligned} & \Phi_{B+B':H+H'}^{A+A':G+G'} \left[\begin{array}{c} (\hat{a}) : (\hat{g}_1); \dots; (\hat{g}_n) \\ (\hat{b}) : (\hat{h}_1); \dots; (\hat{h}_n) \end{array} \middle| \begin{array}{c} \vec{q}; \vec{x} \\ \end{array} \middle| \begin{array}{c} (a') : (g'_1); \dots; (g'_n) \\ (b') : (h'_1); \dots; (h'_n) \end{array} \right] \\ & \equiv \sum_{\vec{m}} \frac{\langle (\hat{a}); q_0 \rangle_m (a')(q_0, m) \prod_{j=1}^n (\langle (\hat{g}_j); q_j \rangle_{m_j} ((g'_j)(q_j, m_j) x_j^{m_j}))}{\langle (\hat{b}); q_0 \rangle_m (b')(q_0, m) \prod_{j=1}^n (\langle (\hat{h}_j); q_j \rangle_{m_j} (h'_j)(q_j, m_j) \langle 1; q_j \rangle_{m_j})} \\ & \quad \times (-1)^{\sum_{j=1}^n m_j (1+H+H'-G-G'+B+B'-A-A')} \\ & \quad \times \text{QE} \left((B+B'-A-A') \binom{m}{2}, q_0 \right) \\ & \quad \times \prod_{j=1}^n \text{QE} \left((1+H+H'-G-G') \binom{m_j}{2}, q_j \right), \end{aligned}$$

where

$$\hat{a} \equiv a \vee \tilde{a} \vee \widetilde{\tilde{a}} \vee_k \tilde{a} \vee \Delta(q; l; \lambda).$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0, m), (g'_j)(q_j, m_j), (b')(q_0, m), (h'_j)(q_j, m_j)$ contain factors of the form $\langle a(\hat{k}); q \rangle_k, (s; q)_k, (s(k); q)_k$ or $\text{QE}(f(\vec{m}))$.

The numbers in front of the colon represent the number of q -shifted factorials with index m in numerator and the denominator. The numbers after the colon denote the number of q -shifted factorials with index m_i in numerator and the denominator. Equally, the numbers after semicolon denote the number of q -shifted factorials with index m_i in numerator and denominator. We can leave out G_2 if it is equal to G_1 for two variables etc. Every ∞ corresponds to multiplication with 1.

Definition 11. The q -derivative is defined by

$$(D_q \varphi)(x) \equiv \begin{cases} \frac{\varphi(x) - \varphi(qx)}{(1-q)x}, & \text{when } q \in \mathbb{C} \setminus \{1\}, x \neq 0; \\ \frac{d\varphi}{dx}(x), & \text{when } q = 1; \\ \frac{d\varphi}{dx}(0), & \text{when } x = 0. \end{cases}$$

Definition 12. Let the Gaussian q -binomial coefficients be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1; q \rangle_n}{\langle 1; q \rangle_k \langle 1; q \rangle_{n-k}}, k = 0, 1, \dots, n.$$

Theorem 2. *The q -binomial theorem:*

$$\sum_{n=0}^{\infty} \frac{\langle a; q \rangle_n}{\langle 1; q \rangle_n} z^n = \frac{(zq^a; q)_{\infty}}{(z; q)_{\infty}},$$

$$|z| < 1, 0 < |q| < 1.$$

Definition 13. If $|q| > 1 \vee 0 < |q| < 1, |z| < |1 - q|^{-1}$, the q -exponential function $E_q(z)$ is defined by

$$(5) \quad E_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k.$$

By the Euler equation (6), the meromorphic continuation of $E_q(z)$ is given by

$$\frac{1}{(z(1-q); q)_{\infty}}.$$

Thus the meromorphic function $\frac{1}{(z(1-q); q)_{\infty}}$, with simple poles at $\frac{q^{-k}}{1-q}, k \in \mathbb{N}$ is a good substitute for $E_q(z)$ in the whole complex plane. We shall however continue to designate this function $E_q(z)$, since it plays an important role in the operator theory.

The q -difference for $E_q(z)$ is

$$D_q E_q(az) = a E_q(az).$$

There is another q -exponential function which is entire when $0 < |q| < 1$ and which converges when $|z| < |1 - q|^{-1}$ if $|q| > 1$. To obtain it, the base in (5) must be inverted, i.e. $q \rightarrow \frac{1}{q}$. This is a common theme in q -calculus.

Definition 14.

$$E_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} z^k.$$

We immediately obtain

$$E_{\frac{1}{q}}(z) = \prod_{n=0}^{\infty} (1 + (1 - q)zq^n), \quad 0 < |q| < 1.$$

The q -difference equation for $E_{\frac{1}{q}}(z)$ is

$$D_q E_{\frac{1}{q}}(az) = a E_{\frac{1}{q}}(qaz),$$

which reduces to the differential equation of the exponential function when q tends to unity.

For later use, we shall need a third q -exponential function:

Definition 15.

$$\widetilde{E}_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty} \frac{(k+1)q^{\binom{k+1}{2}}}{\{k+1\}_q!} z^k.$$

Definition 16. Euler found the following two extra q -analogues of the exponential function:

$$\begin{aligned} e_q(z) &\equiv {}_1\phi_0(\infty; -|q; z) \\ (6) \quad &\equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q \rangle_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1, \quad 0 < |q| < 1. \end{aligned}$$

$$e_{\frac{1}{q}}(z) \equiv {}_0\phi_0(-; -|q; -z) \equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{\langle 1; q \rangle_n} z^n = (-z; q)_{\infty}, \quad 0 < |q| < 1.$$

The second function is an entire function just as the usual exponential function.

Definition 17. We can now define four q -analogues of the trigonometric functions. In the first two equations, $|q| > 1$, or $0 < |q| < 1$ and $|x| < |1 - q|^{-1}$.

$$\begin{aligned} \text{Sin}_q(x) &\equiv \frac{1}{2}(E_q(ix) - E_q(-ix)). \\ \text{Cos}_q(x) &\equiv \frac{1}{2}(E_q(ix) + E_q(-ix)). \end{aligned}$$

$$\text{Sin}_{\frac{1}{q}}(x) \equiv \frac{1}{2}(\text{E}_{\frac{1}{q}}(ix) - \text{E}_{\frac{1}{q}}(-ix)).$$

$$\text{Cos}_{\frac{1}{q}}(x) \equiv \frac{1}{2}(\text{E}_{\frac{1}{q}}(ix) + \text{E}_{\frac{1}{q}}(-ix)),$$

where $x \in \mathbb{C}$ in the last two equations.

Definition 18 ([8]). Three q -Appell function are defined by [9]:

$$\Phi_1(a; b, b'; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1} \langle b'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$\max(|x_1|, |x_2|) < 1.$$

$$\Phi_2(a; b, b'; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1} \langle b'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$$|x_1| \oplus_q |x_2| < 1.$$

$$\Phi_4(a; b; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$$|\sqrt{x_1}| \oplus_q |\sqrt{x_2}| < 1.$$

Remark 1. The function Φ_1 occurs in formulas (33), (36). The function Φ_2 occurs in formulas (31), (43). The function Φ_4 occurs in (30).

Since the number of q -shifted factorials in denominators is larger than in numerators for q -confluent functions, by the quotient criterion, the convergence regions are drastically increased. These convergence regions in the confluent hypergeometric case were only given by Srivastava and Karlsson in [19].

Definition 19.

$$\Psi_1(a; b; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$$|x_1| < 1, |(1 - q)x_2| < \infty,$$

$$\Psi_2(a; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

$$|(1 - q)x_1| < \infty, |(1 - q)x_2| < \infty,$$

$$\Upsilon_1(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$|x_1| < 1, |(1 - q)x_2| < \infty.$$

$$\Upsilon_2(a, a'; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$|(1-q)x_1| < \infty, |(1-q)x_2| < \infty.$$

$$\Upsilon_3(a; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$|(1-q)x_1| < \infty, |(1-q)^2 x_2| < \infty.$$

$$\Xi_1(a, a'; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$|x_1| < 1, |(1-q)x_2| < \infty.$$

$$\Xi_2(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$

$$|x_1| < 1, |(1-q)^2 x_2| < \infty.$$

3. Survey of q -real numbers

The q -real numbers give a convenient notation for q -additions in formal power series, in particular for q -exponential and q -trigonometric functions. There is a one-to-one correspondence between the convergence regions of the two q -Lauricella functions $\Phi_A^{(n)}$ and $\Phi_C^{(n)}$, and the existence of q -real numbers with n letters (or variables).

Definition 20 ([8, p. 24]). Let $a, b \in \mathbb{R}$. Then the NWA q -addition is given by

$$(7) \quad (a \oplus_q b)^n \equiv \sum_{k=0}^n \binom{n}{k}_q a^k b^{n-k}, \quad n = 0, 1, 2, \dots, \quad a \oplus_q b \in \mathbb{R}_{\oplus_q}.$$

In particular, $(a \oplus_q b)^0 \equiv 1$. Furthermore, we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^n \binom{n}{k}_q a^k (-b)^{n-k}, \quad n = 0, 1, 2, \dots$$

Definition 21 ([10]). Let $I^n \subset \mathbb{R}^n, I \equiv (0, 1]$ denote the half-open n -dimensional hypercube. For q fixed, the q -real numbers \mathbb{R}_{\oplus_q} form a subset of the disjoint union of all hypercubes

$$\mathbb{R}_{\oplus_q} \subset \bigcup_{n=2}^{\infty} I^n.$$

For the following definition one could compare with the formula [8, 4.74 p. 110]:

Definition 22. Given $k \in \mathbb{N}$, the formula

$$m_0 + m_1 + \dots + m_j = k$$

determines a set $J_{m_0, \dots, m_j} \in \mathbb{N}^{j+1}$.

Definition 23. For $\vec{m} \in \mathbb{N}^n$ put

$$|\vec{m}| \equiv m_1 + \dots + m_n.$$

If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, its k th NWA-power is given by

$$\left(\oplus_{q, l=0}^{\infty} a_l x^l\right)^k \equiv \left(a_0 \oplus_q a_1 x \oplus_q \dots\right)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0, \dots, m_j}} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q.$$

For $a = (a_1, \dots, a_n) \in I^n$ put

$$\left(a_1 \oplus_q a_2 \oplus_q \dots \oplus_q a_n\right)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_l \in J_{m_0, \dots, m_j}} (a_l)^{m_l} \binom{k}{\vec{m}}_q.$$

Conjecture 1 ([10]). If the function

$$F(k) \equiv \left(a_1 \oplus_q a_2 \oplus_q \dots \oplus_q a_n\right)^k$$

has exactly one absolute maximum in \mathbb{N} , then we have $\lim_{k \rightarrow \infty} F(k) = 0$.

Definition 24. We have $\vec{a} := (a_1, \dots, a_n) \in \mathbb{R}_{\oplus_q}$ exactly when the function $F(k)$ has exactly one absolute maximum.

For the commutative monoid \mathbb{R}_{\oplus_q} we note the following definitions and formulas:

Definition 25. Assume that \sim means equality on $\mathbb{R}[[x]]$ [8, p. 101].

There is a certain linear functional $v : \mathbb{R}[[x]] \times \mathbb{R}_q \mapsto \mathbb{R}$, with $v(f, 0) = a_0 \in \mathbb{R}$, called the evaluation.

Theorem 3. The q -addition (7) has the following properties, for $\alpha, \beta, \gamma \in \mathbb{R}_{\oplus_q}$:

Commutativity:

$$\alpha \oplus_q \beta \sim \beta \oplus_q \alpha.$$

Associativity

$$(\alpha \oplus_q \beta) \oplus_q \gamma \sim \alpha \oplus_q (\beta \oplus_q \gamma).$$

To be able to formulate equation (22), q -Laplace transform of multiplication with $E_q(\alpha t)$, we introduce the following extension of the umbral calculus. Compare with the three formulas in [8, p. 103].

Definition 26. The q -addition $x \uplus_q y$ is defined by another q -Taylor formula:

$$(8) \quad F(x \uplus_q y) \equiv \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} D_{q,x}^n F(q^{-n}x).$$

As before, only positive integer powers of $(x \uplus_q y)$ are used. We call the function argument in (8) \mathbb{R}_{\uplus_q} .

4. On the q -Laplace transform

Several authors have tried to introduce different q -Laplace transforms with upper q -integration limits ∞ and $(s(1-q))^{-1}$. The latter converges to ∞ when $q \rightarrow 1^-$. Also a time-scale approach to this problem has been published by Martin Bohner et al. [1], who used operators similar to our q -real numbers.

This paper was enabled by Erik Koelink and Tom Koornwinder [16], who, in 1992 presented the correct Γ_q function expression as a q -integral with $E_{\frac{1}{q}}(x)$, which enables the correct q -integration by parts proofs. This was possible by using the product expansion for the q -exponential $E_{\frac{1}{q}}(x)$.

Later, in 2005, Kac et al. [3] paid attention to this, and showed how to express the Γ_q function as a q -integral with $E_q(x)$ times an extra factor, a q -analogue of 1. Finally, in 2014, Chung, Kim and Kwon [2] tried to find a q -Laplace transform, which would be useful for practical purposes. We shall now find q -analogues of many properties of the Laplace transform by improving the treatment of the cited paper [2], so as to obtain formulas that do not involve q -integrals of the form \int_0^{∞} , which may be difficult to define. We point out that we shall use the Hahn q -Laplace transform [15], correcting a slight misprint.

4.1. Preliminaries: properties of the q -integral. We first repeat the definitions of q -integrals from [8]. Note that the definitions of these q -integrals for $a, b \in \mathbb{R}_{\oplus_q}$ in [8, (4.80)] are only for umbral use.

Definition 27. The Jackson q -integral is defined by

$$\int_a^b f(t, q) d_q(t) \equiv \int_0^b f(t, q) d_q(t) - \int_0^a f(t, q) d_q(t), \quad a, b \in \mathbb{R},$$

where

$$(9) \quad \int_0^a f(t, q) d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n, \quad 0 < |q| < 1, \quad a \in \mathbb{R}.$$

We now show how the main integral theorems are included in q -analysis.

Definition 28. Let $E_n \equiv \{aq^n\}$, $n = 1, 2, 3, \dots$, $a \in \mathbb{R}^*$ be distinct singleton sets, with respective measures $a(1-q)q^n$. Then the σ -algebra is defined by $\mathcal{M} \equiv \{\mu(E_n) = (1-q)q^n\}_{n=0}^{\infty}$.

By the measure definition, we then have

$$\mu \left(\bigcup_0^{\infty} E_n \right) = \sum_{n=0}^{\infty} \mu(E_n).$$

The q -integral (9) can be written in the form

$$\sum_{n=0}^{\infty} f(aq^n) \mu(E_n),$$

which is the q -integral of f with respect to an infinite discrete measure, that converges weakly to a Lebesgue measure as $q \rightarrow 1^-$.

Put

$$F(x) \equiv \int_0^x f(t, q) d_q(t).$$

Then [5]:

- (1) If $F(x)$ is well-defined, then $D_q f(x) = F(x)$.
- (2) If $f(x)$ is continuous on the closed disk $D(0, r^+)$, then $F(x)$ is well-defined for any $x \in D(0, r^+)$. In fact there exists $K > 0$ such that $|f(q^n x)q^n| \leq K|q|^n$, which guarantees the convergence of the infinite sum.

Let \mathcal{L}_q^1 denote the Banach space of all q -integrable functions on the interval I . The following three theorems are proved analogously to the standard case.

Theorem 4 ([12, p. 52]). *Triangle inequality.*

Let $f(x) \in \mathcal{L}_q^1$. Then

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

Theorem 5. Let $\{f_n\}_0^{\infty}$ be a continuous sequence with limit function

$$(10) \quad f = \lim_{n \rightarrow \infty} f_n,$$

which converges uniformly. Then we have

$$(11) \quad \lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Proof.

$$\begin{aligned} \left| \int f_n d\mu - \int f d\mu \right| &\leq \int |f_n - f| d\mu \leq \int \|f_n - f\| d\mu \\ &= \|f_n - f\| \int d\mu \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

□

Theorem 6. *If the conditions in (10) are satisfied, then we have*

$$\int \sum_{n=0}^{\infty} f_n(x) d\mu = \sum_{n=0}^{\infty} \int f_n(x) d\mu.$$

Proof. Use (11). □

Definition 29. Let Λ_q be the set of all piecewise q -differentiable functions on $(0, \infty)$. We only consider functions in $\mathbb{C}[[z]]$, except the Heaviside step and Dirac functions, which have known q -Laplace transforms.

Definition 30. The equivalence relation \mathcal{A}_q on \mathbb{R}^+ is defined as follows: Elements $a, b > 0$ belong to the same equivalence class,

$$a \sim b \iff \exists n \in \mathbb{Z} : \log \frac{a}{b} = n \log q.$$

The equivalence class $[a], a \in \mathbb{R}^+$ is defined as follows:

$$[a] \equiv \{x \mid \exists n \in \mathbb{Z} : x = aq^n\}.$$

Theorem 7 ([8, p. 204], [14], [17]). *Multiplicative substitution in a q -integral:*

$$(12) \quad \int_0^x f(t, q) d_q(t) = b \int_0^{\frac{x}{b}} f(bt, q) d_q(t).$$

Theorem 8. *Power substitutions $f(x^k, q) \mapsto f(t, q), k \in \mathbb{N}$ in q -integrals:*

$$\int_0^a f(x^k, q) d_q(x) = \frac{1}{\{k\}_q} \int_0^a t^{\frac{1-k}{k}} f(t, q) d_{q^k}(t), \quad a \in \mathbb{R}.$$

Proof. We compute the right hand side.

$$\begin{aligned} & \frac{1-q}{1-q^k} a(1-q^k) \sum_{n=0}^{\infty} q^{(1-k)n} f(aq^{kn}, q) q^{kn} \\ &= a(1-q) \sum_{n=0}^{\infty} f(aq^{kn}, q) q^n. \end{aligned}$$

This equals the left hand side. □

The formulas for substitution in q -integrals above lead to formulas for re-scaling of the measure $\mu(E_n)$, compare with Diaz, Pariguan [4, p. 3]. Assume that $0 < a < b, c > 0$.

For (9) we have :

$$\mu(E_n)[ca, cb] = c\mu(E_n)[a, b].$$

4.2. One-sided q -Laplace transform. This section aims at developing the q -Laplace transform to be used when solving q -difference equations.

We start with the following formula [16]:

$$(13) \quad \Gamma_q(z) = \int_0^{\frac{1}{1-q}} t^{z-1} E_{\frac{1}{q}}(-qt) d_q(t), \quad \operatorname{Re}(z) > 0.$$

We shall now present a corrected version of Hahn [15, (9.1), p. 371].

Theorem 9. *Hahn's definition [15, (9.1), p. 371] does not converge to the Laplace transform when $\lim_{q \rightarrow 1^-}$.*

Proof. Hahn writes in other notation

$$\begin{aligned} \mathcal{L}_q(s) &\equiv \frac{1}{1-q} \int_0^{\frac{1}{s}} f(t) E_{\frac{1}{q}}\left(\frac{-qst}{1-q}\right) d_q(t) \\ &\stackrel{\text{by(12)}}{=} \int_0^{\frac{1}{s(1-q)}} f(t(1-q)) E_{\frac{1}{q}}(-qst) d_q(t), \end{aligned}$$

where we put $b = 1 - q$ in (12). But when $\lim_{q \rightarrow 1^-}$ the function argument in the q -integral converges to 0, which makes no sense. \square

The classical Laplace transform, well known to applied mathematicians and engineers, maps suitable real-valued or complex-valued functions $f(t)$, $t > 0$, to corresponding functions $F(s)$ of another variable s , which are defined for $\operatorname{Re} s > s_0$, where s_0 is function-specific.

When defining the q -Laplace transform, we must use formula (13) and the second q -exponential function $E_{\frac{1}{q}}(x)$, which is entire and has an infinite number of zeros for $x = -\frac{q^n}{1-q}$. The first q -exponential function is not suitable, since it is not entire and has an infinite number of poles. This means that we can only define one q -Laplace transform.

Definition 31. Assume that $\operatorname{Re}(s) > \beta$, and let the function $f \in \Lambda_q$. Then the one-sided q -Laplace transform of f , as a function of s , is defined by

$$(14) \quad \mathcal{L}_q(s) \equiv \mathcal{L}_q(f(t)) \equiv \int_0^{\frac{1}{s(1-q)}} f(t) E_{\frac{1}{q}}(-qst) d_q(t).$$

If f is discontinuous, we divide the q -integral into the corresponding continuous parts.

Formula (14) is perfectly well defined, since

- (1) The upper q -integral limit converges to ∞ for $\lim_{q \rightarrow 1^-}$
- (2) The value of the second q -exponential at the upper q -integral limit is the second zero of $E_{\frac{1}{q}}$.
- (3) We chose the function $E_{\frac{1}{q}}(-qst)$ to get simpler formulas for the q -Laplace transform, after q -integration by parts, which is often used in the proofs.

Remark 2. In the theory of the classical Laplace transform, functions $f(t)$ undergoing transformation are usually required to be ‘of exponential type’, so that the integral defining $F(s)$, which is over the interval $t \in (0, \infty)$, will converge for sufficiently large $\operatorname{Re} s$. The new q -Laplace transform involves a q -integral over a finite interval, from 0 to $\frac{1}{s(1-q)}$; so presumably the function $f(t)$ could fail to be of exponential type, without its q -Laplace transform failing to be defined; though its $\lim_{q \rightarrow 1^-}$ limit would not exist.

Remark 3. A q -Laplace transform $F(s)$ may exist without being defined for all s with $\operatorname{Re} s > 0$. Two examples are formulas (19) and (20).

The one-sided q -Laplace transform (14) has the following properties:

Linearity for q -Laplace transform:

$$(15) \quad \mathcal{L}_q(af(t) + bg(t)) = a\mathcal{L}_q(f(t)) + b\mathcal{L}_q(g(t)), \quad f, g \in \Lambda_q.$$

Proof. This follows from the linearity of the q -integral. \square

We shall compute some q -Laplace transforms, which occur most often.

Theorem 10. *The q -Laplace transform of a power function is given by*

$$(16) \quad \mathcal{L}_q(t^\alpha) = \frac{1}{s^{\alpha+1}} \Gamma_q(\alpha + 1), \quad \alpha \neq -n, \quad n \in \mathbb{N}.$$

Proof. By q -integration by parts, we can show that the function

$$I(s, \alpha) \equiv \int_0^{\frac{1}{s(1-q)}} t^\alpha E_{\frac{1}{q}}(-qst) d_q(t)$$

satisfies the recurrence

$$I(s, \alpha) = \frac{\{\alpha\}_q}{s} I(s, \alpha - 1),$$

which is equivalent to (16). To this end put $u(t, q, \alpha) = t^\alpha$, $v(t, q, s) = -\frac{1}{s} E_{\frac{1}{q}}(-st)$ in [8, (6.58)]. For $E_{\frac{1}{q}}(-qst)$, use the q -derivative formula [8, (6.154)]. \square

Corollary 11. *A q -analogue of [20, (50), p. 163], [6, p. 192]. The one-sided q -Laplace transform of a general q -hypergeometric series times a power function is given by*

$$\begin{aligned} & \mathcal{L}_q(t^{\lambda-1} {}_p\phi_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1} | q; t)) \\ &= \frac{\Gamma_q(\lambda)}{s^\lambda} {}_{p+1}\phi_p(\lambda, a_1, \dots, a_p; b_1, \dots, b_{p-1}, \infty | q; (s(1-q))^{-1}). \end{aligned}$$

Corollary 12. *A q -analogue of [20, (52), p. 164]:*

$$\mathcal{L}_q(t^{\gamma-1} {}_2\phi_1(\alpha, \infty; \gamma | q; t(1-q))) = \frac{\Gamma_q(\gamma)}{s^\gamma} \frac{1}{(\frac{1}{s}; q)_\alpha}.$$

Proof. Use the q -binomial theorem. \square

Theorem 13. Let $H(t - a)$ denote the Heaviside function, where $a \geq 0$:

$$H(t - a) \equiv \begin{cases} 0, & 0 \leq t < a; \\ 1, & t \geq a. \end{cases}$$

The q -Laplace transform of $H(t - a)$ is given by

$$(17) \quad \mathcal{L}_q(H(t - a)) = \frac{1}{s} E_{\frac{1}{q}}(-as).$$

Proof.

$$\int_a^{s^{\frac{1}{1-q}}} E_{\frac{1}{q}}(-qst) d_q(t) = \left[-\frac{1}{s} E_{\frac{1}{q}}(-st) \right]_a^{s^{\frac{1}{1-q}}}.$$

This equals the right hand side, since the upper limit is the first zero of $E_{\frac{1}{q}}$. \square

Theorem 14. Let $\delta(t - a)$ denote the Dirac distribution, where $a \geq 0$. Then the q -Laplace transform of $\delta(t - t_0)$ is given by

$$\mathcal{L}_q(\delta(t - t_0)) = \widetilde{E}_{\frac{1}{q}}(-st_0), \quad t_0 \geq 0.$$

Proof. Put $\delta_a(t - t_0) \equiv \frac{1}{2a} [H(t - (t_0 - a)) - H(t - (t_0 + a))]$. The Dirac distribution can be expressed as

$$\delta(t - t_0) = \lim_{a \rightarrow 0} \delta_a(t - t_0).$$

By linearity and formula (17) this implies

$$(18) \quad \mathcal{L}_q(\delta_a(t - t_0)) = \frac{1}{2a} \left[\frac{1}{s} \left[E_{\frac{1}{q}}(-s(t_0 - a)) - E_{\frac{1}{q}}(-s(t_0 + a)) \right] \right].$$

This is an indeterminate expression $0/0$ and we therefore use L'Hôpital's rule \star :

$$\begin{aligned} \mathcal{L}_q(\delta(t - t_0)) &\stackrel{\text{by(18)}}{=} \lim_{a \rightarrow 0} \frac{1}{2s} \frac{\partial}{\partial a} \left[\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} \left[(as - st_0)^k - (-as - st_0)^k \right] \right] \\ &\stackrel{\text{by}\star}{=} \lim_{a \rightarrow 0} \frac{1}{2} \sum_{k=1}^{\infty} \frac{k \cdot q^{\binom{k}{2}}}{\{k\}_q!} \left[\left[(as - st_0)^{k-1} + (-as - st_0)^{k-1} \right] \right] = \text{RHS}. \quad \square \end{aligned}$$

Lemma 15. *Scaling is given by*

$$\mathcal{L}_q(f(at)) = \frac{1}{a} \mathcal{L}_{q;a} f\left(\frac{s}{a}\right), \quad a > 0.$$

Proof. Use formula (12). □

Theorem 16. *The q -Laplace transform of the q -exponential is given by*

$$(19) \quad \mathcal{L}_q(\mathbb{E}_q(\alpha t)) = \frac{1}{s - \alpha}, \quad \operatorname{Re}\left(\frac{\alpha}{s}\right) < 1.$$

Proof. In the end, the geometric series converges.

$$\begin{aligned} \int_0^{\frac{1}{s(1-q)}} \mathbb{E}_{\frac{1}{q}}(-qst) \mathbb{E}_q(\alpha t) d_q(t) &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\{n\}_q!} \int_0^{\frac{1}{s(1-q)}} \mathbb{E}_{\frac{1}{q}}(-qst) t^n d_q(t) \\ &\stackrel{\text{by(16)}}{=} \sum_{n=0}^{\infty} \frac{\alpha^n}{\{n\}_q!} \frac{\Gamma_q(n+1)}{s^{n+1}} = \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{\alpha}{s}\right)^n = \frac{1}{s} \frac{1}{1 - \frac{\alpha}{s}} = \text{RHS}. \quad \square \end{aligned}$$

Theorem 17. *The q -Laplace transform of the second q -exponential is given by*

$$(20) \quad \mathcal{L}_q\left(\mathbb{E}_{\frac{1}{q}}(\alpha t)\right) = \frac{1}{s} {}_1\phi_1\left(1; \infty|q; -\frac{\alpha}{s}\right), \quad \operatorname{Re}\left(\frac{\alpha}{s}\right) < 1.$$

Proof. Similar to above, the series converges even better. □

Theorem 18. *The q -Laplace transform of q -Sine, where $a \in \mathbb{C}$ and $\operatorname{Re} s > \max[\operatorname{Re}(ia), \operatorname{Re}(-ia)]$ is given by*

$$\mathcal{L}_q(\operatorname{Sin}_q(\alpha t)) = \frac{\alpha}{s^2 + \alpha^2}.$$

The q -Laplace transform of q -Cosine is given by

$$\mathcal{L}_q(\operatorname{Cos}_q(\alpha t)) = \frac{s}{s^2 + \alpha^2}.$$

Proof. Use formulas (15) and (19). □

Theorem 19. *The q -Laplace transform of the n th iterated q -derivative can be expressed as a sum of $D_q^i(f(0^+))$, the i th q -derivative of the function being transformed, evaluated at $t = 0$, or at least evaluated in the limit $t \rightarrow 0^+$:*

$$\mathcal{L}_q(D_q^n(f(t))) = s^n \mathcal{L}_q(f(t)) - \sum_{i=0}^{n-1} s^{n-1-i} D_q^i(f(0^+)), \quad f \in \Lambda_q.$$

Proof. We use q -integration by parts [8, (6.59)].

$$\begin{aligned} \int_0^{\frac{1}{s(1-q)}} E_{\frac{1}{q}}(-qst) D_q^n(f(t)) d_q(t) &= [E_{\frac{1}{q}}(-st) D_q^{n-1}(f(t))]_0^\infty \\ &+ s \int_0^{\frac{1}{s(1-q)}} q(-qst) D_q^{n-1}(f(t)) d_q(t) \\ &= -D_q^{n-1}(f(0)) + s[E_{\frac{1}{q}}(-st) D_q^{n-2}(f(t))]_0^\infty \\ &+ s^2 \int_0^{\frac{1}{s(1-q)}} E_{\frac{1}{q}}(-qst) D_q^{n-2}(f(t)) d_q(t) + \dots = \text{RHS}. \quad \square \end{aligned}$$

Remark 4. Since $f \in \Lambda_q$, this is guaranteed to be finite, except possibly in the discontinuous points.

Corollary 20. *Initial and final value theorems.*

$$\begin{aligned} \lim_{s \rightarrow \infty} s \mathcal{L}_q(f) &= \lim_{t \rightarrow 0^+} f(t), \\ \lim_{s \rightarrow 0} s \mathcal{L}_q(f) &= \lim_{t \rightarrow \infty} f(t). \end{aligned}$$

Proof. Use the previous theorem with $n = 1$. In the first case, the LHS goes to zero when $s \rightarrow \infty$. In the second case, cancel the two terms $f(0^+)$ on each side after letting $s \rightarrow 0^+$. On the LHS $\lim_{s \rightarrow 0^+} \frac{1}{s(1-q)} = +\infty$.

A simpler way is to use the formula (12) for multiplicative substitution in q -integral with the values $x = \frac{1}{s(1-q)}$ and $a = \frac{1}{s}$. \square

Theorem 21. *The q -Laplace transform of multiplication with a power function is given by*

$$(21) \quad \mathcal{L}_q(t^n f(t)) = (-1)^n q^{\binom{n}{2}} D_{q,s}^n \mathcal{L}_q f(q^{-n}s), \quad f \in \Lambda_q.$$

Proof.

$$\begin{aligned} (-1)^n q^{\binom{n}{2}} D_{q,s}^n \mathcal{L}_q(q^{-n}s) &= (-1)^n q^{\binom{n}{2}} \int_0^{\frac{1}{s(1-q)}} D_{q,s}^n (E_{\frac{1}{q}}(-q^{1-n}st)) f(t) d_q(t) \\ &\stackrel{\text{by [8, (6.154)]}}{=} \int_0^{\frac{1}{s(1-q)}} E_{\frac{1}{q}}(-qst) t^n f(t) d_q(t) = \text{LHS}. \quad \square \end{aligned}$$

Theorem 22. *The q -Laplace transform of multiplication with $E_q(\alpha t)$ is given by*

$$(22) \quad \mathcal{L}_q(E_q(\alpha t) f(t)) = \mathcal{L}_q f(s \uplus_q -\alpha), \quad f \in \Lambda_q.$$

Proof.

$$\begin{aligned} \text{LHS} &\stackrel{\text{by (15)}}{=} \sum_{n=0}^{\infty} \frac{\alpha^n}{\{n\}_q!} \mathcal{L}_q(t^n f(t)) \stackrel{\text{by (21)}}{=} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\{n\}_q!} q^{\binom{n}{2}} D_{q,s}^n \mathcal{L}_q f(q^{-n}s) \\ &\stackrel{\text{by (8)}}{=} \text{RHS}. \quad \square \end{aligned}$$

Example 1. We show a simple calculation.

$$\mathcal{L}_q(t\text{Sin}_q(at)) \stackrel{\text{by(21)}}{=} -D_{q,s} \left[\frac{a}{\frac{s^2}{q^2} + a^2} \right] = \frac{\{2\}_q a s}{(s^2 + a^2)(s^2 + q^2 a^2)}.$$

The following table 1 shows the basic q -Laplace transforms.

$f(t)$	$\mathcal{L}_q(s)$
t^α	$\frac{1}{s^{\alpha+1}} \Gamma_q(\alpha + 1)$
$t^{\gamma-1} {}_2\phi_1(\alpha, \infty; \gamma q; t(1-q))$	$\frac{\Gamma_q(\gamma)}{s^\gamma} ((\frac{1}{s}; q)_\alpha)^{-1}$
$H(t-a)$	$s^{-1} E_{\frac{1}{q}}(-as)$
$\delta(t-t_0)$	$\widetilde{E}_{\frac{1}{q}}(-st_0), t_0 \geq 0$
$f(at)$	$\frac{1}{a} \mathcal{L}_{q;a} \left(\frac{s}{a} \right), a > 0$
$E_q(\alpha t)$	$\frac{1}{s-\alpha}, \text{Re} \left(\frac{\alpha}{s} \right) < 1$
$E_{\frac{1}{q}}(\alpha t)$	$\frac{1}{s} {}_1\phi_1 \left(1; \infty q; -\frac{\alpha}{s} \right), \text{Re} \left(\frac{\alpha}{s} \right) < 1$
$\text{Sin}_q(\alpha t)$	$\frac{\alpha}{s^2 + \alpha^2}$
$\text{Cos}_q(\alpha t)$	$\frac{s}{s^2 + \alpha^2}$
$D_q^n(f(t))$	$s^n \mathcal{L}_q(f(t)) - \sum_{i=0}^{n-1} s^{n-1-i} D_q^i(f(0))$
$t^n f(t)$	$(-1)^n q^{\binom{n}{2}} D_{q,s}^n \mathcal{L}_q f(q^{-n}s)$
$E_q(\alpha t) f(t)$	$\mathcal{L}_q(f(s \uplus_q -\alpha))$

TABLE 1. The q -Laplace transforms

4.3. Advanced q -hypergeometric transforms. We continue with some examples of q -Laplace transforms, which are all q -analogues of Exton [11, p. 223–224]. I moved Exton [11, A 6.1.15] and [11, A 6.1.16, p. 224] to the next theorem.

Theorem 23.

$$\begin{aligned}
 (23) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c \end{matrix} \middle| q; xt^2 \right] \right) \\
 & = \frac{1}{s^a} {}_4\phi_3 \left[\begin{matrix} \Delta(q; 2; a) \\ c, 2\infty \end{matrix} \middle| q; \frac{x}{(s(1-q))^2} \right].
 \end{aligned}$$

$$(24) \quad \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} 2\infty \\ a \end{matrix} \middle| q; xt \right] \right) = \frac{1}{s^a} E_q \left(\frac{x}{s(1-q)^2} \right).$$

$$\begin{aligned}
 (25) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c \end{matrix} \middle| q; (x_1 t \oplus_q x_2) \right] \right) \\
 & = \frac{1}{s^a} \Upsilon_3 \left(a; c \middle| q; \frac{x_1}{s(1-q)}, x_2 \right).
 \end{aligned}$$

$$\begin{aligned}
 (26) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} b, \infty \\ c \end{matrix} \middle| q; (x_1 t \oplus_q x_2) \right] \right) \\
 & = \frac{1}{s^a} \Upsilon_1 \left(b, a; c \middle| q; \frac{x_1}{s(1-q)}, x_2 \right).
 \end{aligned}$$

$$\begin{aligned}
 (27) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q (t^{a-1} \Psi_2 (b; c, c' | q; x_1 t, x_2)) \\
 & = \frac{1}{s^a} \Psi_1 \left(b, a; c, c' \middle| q; \frac{x_1}{s(1-q)}, x_2 \right).
 \end{aligned}$$

$$\begin{aligned}
 (28) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c \end{matrix} \middle| q; x_1 t \right] {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c' \end{matrix} \middle| q; x_2 t \right] \right) \\
 & = \frac{1}{s^a} \Psi_2 \left(a; c, c' \middle| q; \frac{x_1}{s(1-q)}, \frac{x_2}{s(1-q)} \right).
 \end{aligned}$$

$$\begin{aligned}
 (29) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c \end{matrix} \middle| q; x_1 t^2 \right] {}_2\phi_1 \left[\begin{matrix} 2\infty \\ c' \end{matrix} \middle| q; x_2 t^2 \right] \right) \\
 & = \frac{1}{s^a} \Phi_{4;4:1}^{4;2} \left[\begin{matrix} \Delta(q; 2; a) : 2\infty; 2\infty \\ 4\infty : c; c' \end{matrix} \middle| q; \frac{x_1}{(s(1-q))^2}, \frac{x_2}{(s(1-q))^2} \right].
 \end{aligned}$$

$$\begin{aligned}
 (30) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q (t^{a-1} \Psi_2 (b; c, c' | q; x_1 t, x_2 t)) \\
 & = \frac{1}{s^a} \Phi_4 \left(a, b; c, c' \middle| q; \frac{x_1}{s(1-q)}, \frac{x_2}{s(1-q)} \right).
 \end{aligned}$$

$$\begin{aligned}
 (31) \quad & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} {}_2\phi_1 \left[\begin{matrix} b, \infty \\ c \end{matrix} \middle| q; x_1 t \right] {}_2\phi_1 \left[\begin{matrix} b', \infty \\ c' \end{matrix} \middle| q; x_2 t \right] \right) \\
 & = \frac{1}{s^a} \Phi_2 \left(a, b, b'; c, c' \middle| q; \frac{x_1}{s(1-q)}, \frac{x_2}{s(1-q)} \right).
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} \Psi_2 (b; a, c | q; x_1 t, x_2) \right) \\
 (32) \quad & = \frac{1}{s^a \left(\frac{x_1}{s(1-q)}; q \right)_b} {}_3\phi_2 \left[\begin{matrix} b, 2\infty \\ c \end{matrix} \middle| q; x_2 \right] \left(\frac{x_1}{s(1-q)} q^b; q \right)_k \left[\begin{matrix} \cdot \\ \cdot \end{matrix} \right], \\
 & \left| \frac{x_1}{s(1-q)} \right| < 1.
 \end{aligned}$$

Proof. First we prove (25). The left hand side equals

$$\begin{aligned}
 & \frac{1}{\Gamma_q(a)} \int_0^{\frac{1}{s(1-q)}} t^{a-1} E_{\frac{1}{q}}(-qst) \sum_{m_1, m_2=0}^{\infty} \frac{(x_1 t)^{m_2} x_2^{m_1-m_2}}{\langle c; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle 1; q \rangle_{m_1-m_2}} d_q(t) \\
 & \stackrel{\text{by(16)}}{=} \sum_{m_1, m_2=0}^{\infty} \Gamma_q \left[\begin{matrix} a + m_2 \\ a \end{matrix} \right] \frac{x_1^{m_2} x_2^{m_1-m_2}}{\langle 1; q \rangle_{m_2} \langle 1; q \rangle_{m_1-m_2} \langle c; q \rangle_{m_1} s^{a+m_2}} \stackrel{\text{by[8, (1.46)]}}{=} \text{RHS.}
 \end{aligned}$$

Then we prove formula (32). The left hand side equals

$$\begin{aligned}
 & \frac{1}{\Gamma_q(a)} \int_0^{\frac{1}{s(1-q)}} t^{a-1} E_{\frac{1}{q}}(-qst) \sum_{m_1, m_2=0}^{\infty} \frac{\langle b; q \rangle_{m_1+m_2}}{\langle a, 1; q \rangle_{m_1} \langle c, 1; q \rangle_{m_2}} (x_1 t)^{m_1} x_2^{m_2} d_q(t) \\
 & \stackrel{\text{by(16)}}{=} \sum_{m_1, m_2=0}^{\infty} \Gamma_q \left[\begin{matrix} a + m_1 \\ a \end{matrix} \right] \frac{\langle b; q \rangle_{m_1+m_2}}{\langle a, 1; q \rangle_{m_1} \langle c, 1; q \rangle_{m_2} s^{a+m_1}} x_1^{m_1} x_2^{m_2} \\
 & \stackrel{\text{by[8, (1.46)]}}{=} \frac{1}{s^a} \sum_{m_2=0}^{\infty} \frac{\langle b; q \rangle_{m_2}}{\langle c, 1; q \rangle_{m_2}} x_2^{m_2} \\
 & \quad \times \sum_{m_1=0}^{\infty} \frac{\langle b + m_2 \rangle_{m_1}}{\langle 1; q \rangle_{m_1}} \left(\frac{x_1}{s(1-q)} \right)^{m_1} \stackrel{\text{by[8, (7.27)]}}{=} \text{RHS.} \quad \square
 \end{aligned}$$

The following formulas are all q -analogues of Erdélyi [7].

Theorem 24. For $\text{Re}(b') > 0$, $\text{Re}(s) > 0$, $\text{Re}(s) > \text{Re}(|y|)$, we have a q -analogue of Erdélyi [7, 1, p. 222]:

$$\begin{aligned}
 (33) \quad & \frac{1}{\Gamma_q(b')} \mathcal{L}_q \left(t^{b'-1} \Upsilon_1 (a; b; c | q; x, yt) \right) \\
 & = \frac{1}{s^{b'}} \Phi_1 \left(a, b, b'; c \middle| q; x, \frac{y}{s(1-q)} \right).
 \end{aligned}$$

For $\text{Re}(b) > 0$, $\text{Re}(s) > 0$, $\text{Re}(s) > \text{Re}(|x|)$, we have a q -analogue of Erdélyi [7, 2, p. 222]:

$$\begin{aligned}
 (34) \quad & \frac{1}{\Gamma_q(b)} \mathcal{L}_q \left(t^{b-1} \Upsilon_2 (a, a'; c | q; xt, y) \right) \\
 & = \frac{1}{s^b} \Xi_1 \left(a, a', b; c \middle| q; \frac{x}{s(1-q)}, y \right).
 \end{aligned}$$

For $\operatorname{Re}(c) > 0$, $\operatorname{Re}(s) > \max(\operatorname{Re}(|x|), \operatorname{Re}(|y|))$, we have a q -analogue of Erdélyi [7, 3, p. 222]:

$$(35) \quad \frac{1}{\Gamma_q(c)} \mathcal{L}_q (t^{c-1} \Upsilon_2 (b, b'; c|q; xt, y)) = \frac{1}{s^c \left(\frac{x}{(s(1-q))}; q \right)_b \left(\frac{y}{(s(1-q))}; q \right)_{b'}}.$$

For $\operatorname{Re}(a) > 0$, $\operatorname{Re}(s) > \max(\operatorname{Re}(|x|), \operatorname{Re}(|y|))$, we have a q -analogue of Erdélyi [7, 4, p. 222]:

$$(36) \quad \begin{aligned} & \frac{1}{\Gamma_q(a)} \mathcal{L}_q (t^{a-1} \Upsilon_2 (b, b'; c|q; xt, yt)) \\ &= \frac{1}{s^a} \Phi_1 \left(a, b, b'; c \left| q; \frac{x}{s(1-q)}, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\operatorname{Re}(c) > 0$, $\operatorname{Re}(s) > \max(\operatorname{Re}(|x_i|), i = 1, \dots, n)$, we have a q -analogue of Erdélyi [7, 5, p. 222] for the Humbert function:

$$(37) \quad \frac{1}{\Gamma_q(c)} \mathcal{L}_q (t^{c-1} \Upsilon_2 (\vec{b}; c|q; \vec{x}t)) = \frac{1}{s^c \left(\frac{\vec{x}}{(s(1-q))}; q \right)_{\vec{b}}}.$$

For $\operatorname{Re}(a) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|x|)$, we have a q -analogue of Erdélyi [7, 6, p. 222]:

$$(38) \quad \frac{1}{\Gamma_q(a)} \mathcal{L}_q (t^{a-1} \Upsilon_3 (b; c|q; tx, y)) = \frac{1}{s^a} \Xi_2 \left(a, b; c \left| q; \frac{x}{s(1-q)}, y \right. \right).$$

For $\operatorname{Re}(b') > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|y|)$, we have a q -analogue of Erdélyi [7, 7, p. 223]:

$$(39) \quad \frac{1}{\Gamma_q(b')} \mathcal{L}_q (t^{b'-1} \Upsilon_3 (b; c|q; x, yt)) = \frac{1}{s^{b'}} \Upsilon_2 \left(b, b'; c \left| q; x, \frac{y}{s(1-q)} \right. \right).$$

For $\operatorname{Re}(a) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \frac{\sqrt{y}}{1-q}$, we have a q -analogue of Erdélyi [7, 8, p. 223]:

$$(40) \quad \begin{aligned} & \frac{1}{\Gamma_q(2a)} \mathcal{L}_q (t^{2a-1} \Upsilon_3 (b; c|q; x, yt^2)) \\ &= \frac{1}{s^{2a}} \Phi_{1:3;0}^{1:4;1} \left[\begin{array}{c} \infty : \Delta(q; 2; 2a) \\ c : 3\infty \end{array} ; b \left| q; \frac{y}{(s(1-q))^2}, x \right. \right]. \end{aligned}$$

For $\operatorname{Re}(c) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|x|)$, we have a q -analogue of Erdélyi [7, 9, p. 223]:

$$(41) \quad \frac{1}{\Gamma_q(c)} \mathcal{L}_q (t^{c-1} \Upsilon_3 (b; c|q; xt, yt)) = \frac{1}{s^c \left(\frac{x}{(s(1-q))}; q \right)_b} E_q \left(\frac{y}{s(1-q)^2} \right).$$

For $\operatorname{Re}(a) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \max(\operatorname{Re}(|x|), \operatorname{Re}(|y|))$, we have a q -analogue of Erdélyi [7, 10, p. 223]:

$$(42) \quad \begin{aligned} & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} \Upsilon_3 (b; c|q; xt, yt) \right) \\ &= \frac{1}{s^a} \Upsilon_1 \left(a, b; c \left| q; \frac{x}{s(1-q)}, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\operatorname{Re}(b') > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|y|)$, we have a q -analogue of Erdélyi [7, 11, p. 223]:

$$(43) \quad \begin{aligned} & \frac{1}{\Gamma_q(b')} \mathcal{L}_q \left(t^{b'-1} \Psi_1 (a, b; c, c'|q; x, yt) \right) \\ &= \frac{1}{s^{b'}} \Phi_2 \left(a, b, b'; c, c' \left| q; x, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\operatorname{Re}(b) > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|x|)$, we have a q -analogue of Erdélyi [7, 12, p. 223]:

$$(44) \quad \begin{aligned} & \frac{1}{\Gamma_q(b)} \mathcal{L}_q \left(t^{b-1} \Psi_2 (a; c, c'|q; x, yt) \right) \\ &= \frac{1}{s^{b'}} \Psi_1 \left(a, b; c, c' \left| q; \frac{x}{s(1-q)}, y \right. \right). \end{aligned}$$

For $\operatorname{Re}(a) > 0$, $\operatorname{Re}(s) > \max(\operatorname{Re}(|x|), \operatorname{Re}(|y|))$, we have a q -analogue of Erdélyi [7, 13, p. 223]:

$$(45) \quad \begin{aligned} & \frac{1}{\Gamma_q(a)} \mathcal{L}_q \left(t^{a-1} \Psi_2 (b; c, c'|q; xt, yt) \right) \\ &= \frac{1}{s^a} \Phi_4 \left(a, b; c, c' \left| q; \frac{x}{s(1-q)}, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\operatorname{Re}(b') > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|y|)$, we have a q -analogue of Erdélyi [7, 14, p. 223]:

$$(46) \quad \begin{aligned} & \frac{1}{\Gamma_q(b')} \mathcal{L}_q \left(t^{b'-1} \Xi_1 (a, a'; b; c|q; x, yt) \right) \\ &= \frac{1}{s^{b'}} \Phi_3 \left(a, a', b, b'; c \left| q; x, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\operatorname{Re}(a') > 0$, $\operatorname{Re}(s) > 0$, $\operatorname{Re}(s) > \operatorname{Re}(|y|)$, we have a q -analogue of Erdélyi [7, 15, p. 223]:

$$(47) \quad \begin{aligned} & \frac{1}{\Gamma_q(a')} \mathcal{L}_q \left(t^{a'-1} \Xi_2 (a, b; c|q; x, yt) \right) \\ &= \frac{1}{s^{a'}} \Xi_1 \left(a, a'; b; c \left| q; x, \frac{y}{s(1-q)} \right. \right). \end{aligned}$$

For $\text{Re}(a') > 0$, $\text{Re}(s) > 0$, $\text{Re}(s) > 2\text{Re}(\sqrt{y})$, we have a q -analogue of Erdélyi [7, 16, p. 223]:

$$(48) \quad \frac{1}{\Gamma_q(2a')} \mathcal{L}_q \left(t^{2a'-1} \Xi_2(a, b; c | q; x, yt^2) \right) \\ = \frac{1}{s^{2a'}} \Phi_{1:2;4}^{1:1;3} \left[\begin{matrix} \infty : a, b, & \Delta(q; 2; 2a') \\ c : \infty; & 3\infty \end{matrix} \middle| q; x, \frac{y}{(s(1-q))^2} \right].$$

Proof. First we prove (40). The left hand side equals

$$\frac{1}{\Gamma_q(2a)} \int_0^{s(1-q)^{-1}} t^{2a-1} E_{\frac{1}{q}}(-qst) \sum_{m_1, m_2=0}^{\infty} \frac{\langle b; q \rangle_{m_1} x^{m_1} (yt^2)^{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} d_q(t) \\ \stackrel{\text{by(16)}}{=} \frac{1}{s^{2a}} \sum_{m_1, m_2=0}^{\infty} \Gamma_q \left[\begin{matrix} 2a + 2m_2 \\ 2a \end{matrix} \right] \frac{\langle b; q \rangle_{m_1} x^{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} \left(\frac{y}{s^2} \right)^{m_2} \\ \stackrel{\text{by[8, (1.46)]}}{=} \frac{1}{s^{2a}} \sum_{m_1, m_2=0}^{\infty} \frac{\langle b; q \rangle_{m_1} \langle 2a; q \rangle_{2m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x^{m_1} \left(\frac{y}{(s(1-q))^2} \right)^{m_2} = \text{RHS.}$$

Then we prove (41). The left hand side equals

$$\frac{1}{\Gamma_q(c)} \int_0^{s(1-q)^{-1}} t^{c-1} E_{\frac{1}{q}}(-qst) \sum_{m_1, m_2=0}^{\infty} \frac{\langle b; q \rangle_{m_1} (xt)^{m_1} (yt)^{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} d_q(t) \\ \stackrel{\text{by(16)}}{=} \frac{1}{s^c} \sum_{m_1, m_2=0}^{\infty} \Gamma_q \left[\begin{matrix} c + m_1 + m_2 \\ c \end{matrix} \right] \frac{\langle b; q \rangle_{m_1} x^{m_1} y^{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2} s^{m_1+m_2}} \\ \stackrel{\text{by[8, (1.46)]}}{=} \frac{1}{s^c} \sum_{m_2=0}^{\infty} \left(\frac{y}{s(1-q)} \right)^{m_2} \frac{1}{\langle 1; q \rangle_{m_2}} \\ \times \sum_{m_1=0}^{\infty} \left(\frac{x}{s(1-q)} \right)^{m_1} \frac{\langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1}} \stackrel{\text{by[8, (7.27)]}}{=} \text{RHS.} \quad \square$$

5. Conclusion

The table of q -Laplace transforms enables us to quickly find out which formula to be used. We are thus ready to solve inhomogenous q -difference equations, with right hand side for instance a delta function. The solutions will be the sum of the homogenous and the inhomogenous solutions like for differential equations. In the next paper we will solve the corresponding system of q -difference equations.

6. Discussion

It was not possible to find a q -analogue of the transform for $f(t-a)H(t-a)$, since an additive substitution in q -integrals is not allowed. In the next step

we could consider multiple q -Laplace transforms, which would then be q -analogues of well-known multiple Laplace transforms. The proof of the q -analogue of the Bromwich integral would require the corresponding Cauchy integral formula. We shall, however, try to discuss inversion problems in certain special cases in the next paper.

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