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The Euler-like operators on tuples of Lagrangians and functions on bases

ABSTRACT. Let $\mathcal{FM}_{m,n}$ denote the category of fibered manifolds with m -dimensional bases and n -dimensional fibres and their fibered diffeomorphisms onto open images. We describe all $\mathcal{FM}_{m,n}$ -natural operators C transforming tuples (λ, g) of Lagrangians $\lambda : J^s Y \rightarrow \bigwedge^m T^*M$ (or formal Lagrangians $\lambda : J^s Y \rightarrow V^*J^s Y \otimes \bigwedge^m T^*M$) on $\mathcal{FM}_{m,n}$ -objects $Y \rightarrow M$ and functions $g : M \rightarrow \mathbf{R}$ into Euler maps $C(\lambda, g) : J^{2s} Y \rightarrow V^*Y \otimes \bigwedge^m T^*M$ on Y . The most important example of such C is the Euler operator E (from the variational calculus) (or the formal Euler operator \mathbf{E}) treated as the operator in question depending only on Lagrangians (or formal Lagrangians).

1. Introduction. All manifolds considered in this paper are assumed to be finite dimensional, without boundary and smooth (i.e. of class \mathcal{C}^∞). Mappings between manifolds are assumed to be smooth (of class \mathcal{C}^∞).

Given a fibred manifold $Y \rightarrow M$, we have the s -jet prolongation $J^s Y$ of $Y \rightarrow M$ and the obvious jet projection $\pi_s^{2s} : J^{2s} Y \rightarrow J^s Y$ for any positive integer s . We also have the vertical bundle $VY \rightarrow Y$, its dual bundle $V^*Y \rightarrow Y$, the cotangent bundle T^*M and its m th inner product $\bigwedge^m T^*M$. Given fibred manifolds $Z_1 \rightarrow M$ and $Z_2 \rightarrow M$ with the same basis M , let $\mathcal{C}_M^\infty(Z_1, Z_2)$ denote the space of all base preserving fibred maps of Z_1 into Z_2 . Let m be the dimension of the base M of Y . Elements from the space $\mathcal{C}_M^\infty(J^s Y, \bigwedge^m T^*M)$ are called (sth order) Lagrangians on $Y \rightarrow M$. Elements

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from the space $\mathcal{C}_{J^s Y}^\infty(J^s Y, V^* J^s Y \otimes \bigwedge^m T^* M)$ are called (sth order) formal Lagrangians on $Y \rightarrow M$. Elements from the space $\mathcal{C}_Y^\infty(J^q Y, V^* Y \otimes \bigwedge^m T^* M)$ are called Euler maps on $Y \rightarrow M$. The concept of natural operators can be found in [3].

By Proposition 49.3 of [3], any sth order Lagrangian $\lambda : J^s Y \rightarrow \bigwedge^m T^* M$ on a fibered manifold $Y \rightarrow M$ induces canonically the Euler map $E(\lambda) : J^{2s} Y \rightarrow V^* Y \otimes \bigwedge^m T^* M$. So, we have the so-called Euler operator

$$E : \mathcal{C}_M^\infty\left(J^s Y, \bigwedge^m T^* M\right) \rightarrow \mathcal{C}_Y^\infty\left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M\right).$$

In [1] (see [4]), I. Kolář proved the following:

Theorem 1.1. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural operator*

$$C : \mathcal{C}_M^\infty\left(J^s Y, \bigwedge^m T^* M\right) \rightarrow \mathcal{C}_Y^\infty\left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M\right)$$

is of the form cE , $c \in \mathbf{R}$, where E is the Euler operator.

Here and later $\mathcal{FM}_{m,n}$ denotes the category of fibered manifolds with m -dimensional bases and n -dimensional fibres and their fibered diffeomorphisms onto open images.

In our paper, we study the more general problem how a tuple (λ, g) of a Lagrangian $\lambda \in \mathcal{C}_M^\infty(J^s Y, \bigwedge^m T^* M)$ on an $\mathcal{FM}_{m,n}$ -object $Y \rightarrow M$ and a map $g \in \mathcal{C}^\infty(M, \mathbf{R})$ can induce canonically an Euler map $C(\lambda, g) \in \mathcal{C}_Y^\infty(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M)$.

Namely, in our paper, if $m \geq 2$, we describe all regular and π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural operators

$$C : \mathcal{C}_M^\infty\left(J^s Y, \bigwedge^m T^* M\right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty\left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M\right).$$

Further, in [2] (see [4]), I. Kolář introduced the so-called formal Euler operator

$$\mathbf{E} : \mathcal{C}_{J^s Y}^\infty\left(J^s Y, V^* J^s Y \otimes \bigwedge^m T^* M\right) \rightarrow \mathcal{C}_Y^\infty\left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M\right)$$

for all $\mathcal{FM}_{m,n}$ -objects $Y \rightarrow M$. In [6], we proved:

Theorem 1.2. *Let m, n, s be positive integers. Then any regular, π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural operator*

$$C : \mathcal{C}_{J^s Y}^\infty\left(J^s Y, V^* J^s Y \otimes \bigwedge^m T^* M\right) \rightarrow \mathcal{C}_Y^\infty\left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M\right)$$

is of the form $c\mathbf{E}$, $c \in \mathbf{R}$, where \mathbf{E} is the formal Euler operator.

In our paper, if $m \geq 2$, we also describe all regular, π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural operators

$$C : \mathcal{C}_{J^s Y}^\infty \left(J^s Y, V^* J^s Y \otimes \bigwedge^m T^* M \right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty \left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M \right).$$

2. The main results.

Example 2.1. Let $l=0, 1, \dots, s$. We define $E^{(l)}(\lambda, g) : J^{2s} Y \rightarrow V^* Y \otimes \bigwedge^m T^* M$ by

$$E^{(l)}(\lambda, g)|_{j_{x_o}^{2s} \sigma} := E((-1)^l (g - g(x_o))^l \cdot \lambda)|_{j_{x_o}^{2s} \sigma}$$

for any $\lambda \in \mathcal{C}_M^\infty(J^s Y, \bigwedge^m T^* M)$ on an $\mathcal{FM}_{m,n}$ -object $Y \rightarrow M$, any $g \in \mathcal{C}^\infty(M, \mathbf{R})$, any $j_{x_o}^{2s} \sigma \in J_{x_o}^{2s} Y$ and any $x_o \in M$, where E is the Euler operator. Thus we have the corresponding $\mathcal{FM}_{m,n}$ -natural operator

$$E^{(l)} : \mathcal{C}_M^\infty \left(J^s Y, \bigwedge^m T^* M \right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty \left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M \right).$$

We call $E^{(l)}$ the l th modification of E . Clearly, $E^{(0)} = E$.

The first main result of our paper is the following:

Theorem 2.2. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural (i.e. invariant with respect to $\mathcal{FM}_{m,n}$ -maps) operator*

$$C : \mathcal{C}_M^\infty \left(J^s Y, \bigwedge^m T^* M \right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty \left(J^{2s} Y, V^* Y \otimes \bigwedge^m T^* M \right)$$

is $C = \sum_{l=0}^s h_l \cdot E^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \rightarrow \mathbf{R}$, $l = 0, \dots, s$, where the multiplication $h \cdot C$ is defined by

$$(h \cdot C)(\lambda, g)|_{j_{x_o}^{2s} \sigma} = h(g(x_o)) \cdot C(\lambda, g)|_{j_{x_o}^{2s} \sigma}$$

for any $h : \mathbf{R} \rightarrow \mathbf{R}$, any C in question and any $\lambda, g, j_{x_o}^{2s} \sigma$ as above.

In other words, the space of all C in question is the free $(s+1)$ -dimensional $\mathcal{C}^\infty(\mathbf{R})$ -module and the operators $E^{(l)}$ for $l=0, 1, \dots, s$ form the basis in this module.

Remark 2.3. The $\mathcal{FM}_{m,n}$ -invariance of C means that for any $\mathcal{FM}_{m,n}$ -map $f : Y \rightarrow Y_1$, Lagrangians $\lambda \in \mathcal{C}_M^\infty(J^s Y, \bigwedge^m T^* M)$, $\lambda_1 \in \mathcal{C}_{M_1}^\infty(J^s Y_1, \bigwedge^m T^* M_1)$ and maps $g : M \rightarrow \mathbf{R}$ and $g_1 : M_1 \rightarrow \mathbf{R}$, if λ and λ_1 are f -related and g and g_1 are f -related, then so are $C(\lambda)$ and $C(\lambda_1)$. The π_s^{2s} -locality of C means that $C(\lambda, g)_\rho$ depends on $\text{germ}_{\pi_s^{2s}(\rho)}(\lambda, g)$ for any $\rho \in J^{2s} Y$ and $\lambda \in \mathcal{C}_M^\infty(J^s Y, \bigwedge^m T^* M)$ and any $g \in \mathcal{C}^\infty(M, \mathbf{R})$. The regularity of C means that C transforms smoothly parametrized families of tuples of Lagrangians and maps on the bases into smoothly parametrized families of Euler maps.

Clearly, Theorem 1.1 is a simple consequence of Theorem 2.2. The proof of Theorem 2.2 will be given in Section 4.

Remark 2.4. If $m = 1$, Theorem 2.2 does not hold. Indeed, in [1], I. Kolář constructed a regular, π_1^2 -local and $\mathcal{FM}_{1,n}$ -natural operator $W : \mathcal{C}_M^\infty(J^1Y, T^*M) \rightarrow \mathcal{C}_Y^\infty(J^2Y, V^*Y \otimes \bigwedge^m T^*M)$ which is not cE . Suppose $W = \sum_{l=0}^s h_l \cdot E^{(l)}$. Then $W(\lambda) = W(\lambda, 1) = h_0(1)E(\lambda)$, i.e. $W = h_0(1)E$ which is a contradiction.

Example 2.5. Let $l = 0, \dots, s$. By the same way as in the previous example, we define $\mathbf{E}^{(l)}(\lambda, g) : J^{2s}Y \rightarrow V^*Y \otimes \bigwedge^m T^*M$ by

$$\mathbf{E}^{(l)}(\lambda, g)|_{j_{x_o}^{2s}\sigma} := \mathbf{E}((-1)^l (g - g(x_o))^l \cdot \lambda)|_{j_{x_o}^{2s}\sigma}$$

for any $\lambda \in \mathcal{C}_M^\infty(J^sY, V^*J^sY \otimes \bigwedge^m T^*M)$, any $g \in C^\infty(M, \mathbf{R})$, any $j_{x_o}^{2s}\sigma \in J_{x_o}^{2s}Y$ and any $x_o \in M$, where \mathbf{E} is the formal Euler operator. Thus we have the corresponding $\mathcal{FM}_{m,n}$ -natural operator

$$\mathbf{E}^{(l)} : \mathcal{C}_{J^sY}^\infty \left(J^sY, V^*J^sY \otimes \bigwedge^m T^*M \right) \times C^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty \left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M \right).$$

We call $\mathbf{E}^{(l)}$ the l th modification of \mathbf{E} . Clearly, $\mathbf{E}^{(0)} = \mathbf{E}$.

The second main result of our paper is the following:

Theorem 2.6. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π_s^{2s} -local and $\mathcal{FM}_{m,n}$ -natural operator*

$$C : \mathcal{C}_{J^sY}^\infty \left(J^sY, V^*J^sY \otimes \bigwedge^m T^*M \right) \times C^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_Y^\infty \left(J^{2s}Y, V^*Y \otimes \bigwedge^m T^*M \right)$$

is $C = \sum_{l=0}^s h_l \cdot \mathbf{E}^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \rightarrow \mathbf{R}$, $l = 0, \dots, s$, where $h \cdot C$ is quite similar as in the previous theorem.

So, the space of all C in question is also the free $(s+1)$ -dimensional $C^\infty(\mathbf{R})$ -module and the operators $\mathbf{E}^{(l)}$ for $l = 0, 1, \dots, s$ form the basis in this module.

Clearly, Theorem 1.2 for $m \geq 2$ is a simple consequence of Theorem 2.6. The schema of the proof of Theorem 2.6 will be presented in Section 5.

3. Preparation. From now on, let $\mathbf{N} = \{0, 1, 2, \dots\}$ be the set of non-negative integers and let $\mathbf{R}^{m,n}$ denotes the trivial (affine) bundle $\mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}^m$ and let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$. Let $dx^\mu = dx^1 \wedge \dots \wedge dx^m$. Given $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$, let $x^\alpha := (x^1)^{\alpha_1} \dots (x^m)^{\alpha_m}$. Given $i = 1, \dots, m$, let $1_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{N}^m$ where 1 is in the i th position.

Given a fibred manifold $Y \rightarrow M$, the s -jet prolongation J^sY of $Y \rightarrow M$ is the fibred manifold (with the base M) of all s -jets $j_x^s\sigma$ at $x \in M$ of local sections $\sigma : M \rightarrow Y$ of $Y \rightarrow M$. If $Y \rightarrow M$ and $Y^1 \rightarrow M_1$ are fibred manifolds with m -dimensional bases M and M_1 and $f : Y \rightarrow Y^1$ is a fibred map with the base map $\underline{f} : M \rightarrow M_1$ being local diffeomorphism, then we have the fibred

map $J^s f: J^s Y \rightarrow J^s Y_1$ defined by $J^r f(j_{x_o}^s \sigma) = j_{\underline{f}(x_o)}^s (f \circ \sigma \circ \underline{f}^{-1})$, $j_{x_o}^s \sigma \in J_{x_o}^s Y$, $x_o \in M$.

Let $((x^i), (y_\alpha^j))$ be the induced coordinates on $J^s(\mathbf{R}^{m,n})$, where $i=1, \dots, m$, $j=1, \dots, n$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$ are such that $|\alpha| = \alpha_1 + \dots + \alpha_m \leq s$. We remind that

$$x^i(j_{x_o}^s \sigma) = x_o^i \quad \text{and} \quad y_\alpha^j(j_{x_o}^s \sigma) = (\partial_\alpha \sigma^j)(x_o)$$

for any $j_{x_o}^s \sigma = j_{x_o}^s(\sigma^1, \dots, \sigma^n) \in J_{x_o}^s(\mathbf{R}^{m,n}) = J_{x_o}^s(\mathbf{R}^m, \mathbf{R}^n)$, $x_o \in \mathbf{R}^m$, where ∂_α is the iterated partial derivative as indicated multiplied by $\frac{1}{\alpha!}$.

Lemma 3.1. *Let $i=1, \dots, m$, $j=1, \dots, n$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$ be such that $|\alpha| \leq s$.*

(i) *For any $\tau = (\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we have*

$$(J^s \psi_\tau)_* y_\alpha^j = \tau^j y_\alpha^j,$$

where $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1} y^1, \dots, \frac{1}{\tau^n} y^n)$ is the $\mathcal{FM}_{m,n}$ -map.

(ii) *For any $t \in \mathbf{R} \setminus \{0\}$, we have*

$$(J^s \varphi_t^i)_* y_\alpha^j = t^{-\alpha_i} y_\alpha^j,$$

where $\varphi_t^i = (x^1, \dots, \frac{1}{t} x^i, \dots, x^m, y^1, \dots, y^n)$ is the $\mathcal{FM}_{m,n}$ -map.

(iii) *If $\alpha_i \neq 0$, we have*

$$(J^s \psi^{(i)})_* y_\alpha^1 = y_\alpha^1 + x^i y_\alpha^1 + y_{\alpha-1_i}^1,$$

where $\psi^{(i)} = (x^1, \dots, x^m, y^1 + x^i y^1, y^2, \dots, y^n)^{-1}$ is the $\mathcal{FM}_{m,n}$ -map (defined over $0 \in \mathbf{R}^m$).

(iv) *If $\alpha_1 \neq 0$, we have*

$$(J^s \chi_t)_* y_{\alpha-1_1+1_2}^1 = y_{\alpha-1_1+1_2}^1 + c_1 t y_\alpha^1 + \dots + c_{\alpha_2+1} t^{\alpha_2+1} y_{(\alpha_1+\alpha_2, 0, \alpha_3, \dots, \alpha_m)}^1$$

for some $c_1, \dots \in \mathbf{R}$ with $c_1 \neq 0$, where $\chi_t = (x^1 + t x^2, x^2, \dots, x^m, y^1, \dots, y^n)$ is the $\mathcal{FM}_{m,n}$ -map (defined if $m \geq 2$).

Proof. We prove (iii) and (iv), only. The proofs of the other parts are similar.

Ad (iii). If $\alpha_i \neq 0$, we have

$$\begin{aligned} y_\alpha^1 \circ J^s(\psi^{(i)})^{-1}(j_{x_o}^s \sigma) &= \partial_\alpha(\sigma^1 + x^i \sigma^1)(x_o) \\ &= \partial_\alpha \sigma^1(x_o) + x_o^i \partial_\alpha \sigma^1(x_o) + \partial_{\alpha-1_i} \sigma^1(x_o) \end{aligned}$$

for any $j_{x_o}^s \sigma \in J^s(\mathbf{R}^{m,n}) = J^s(\mathbf{R}^m, \mathbf{R}^n)$. Then

$$((J^s \psi^{(i)})_* y_\alpha^1)(j_{x_o}^s \sigma) = y_\alpha^1 \circ J^s(\psi^{(i)})^{-1}(j_{x_o}^s \sigma) = (y_\alpha^1 + x^i y_\alpha^1 + y_{\alpha-1_i}^1)(j_{x_o}^s \sigma)$$

for any $j_{x_o}^s \sigma \in J^s(\mathbf{R}^{m,n}) = J^s(\mathbf{R}^m, \mathbf{R}^n)$.

Ad (iv). Similarly, we have

$$\begin{aligned} y_{\alpha-1+1_2}^1 \circ (J^s \chi_t)^{-1} (j_{x_o}^s \sigma) &= \partial_{\alpha-1+1_2} (\sigma^1(x^1 + tx^2, x^2, \dots, x^m)) ((\chi_t)^{-1}(x_o)) \\ &= \partial_{\alpha-1+1_2} \sigma^1(x_o) + c_1 t \partial_{\alpha} \sigma^1(x_o) + c_2 t^2 \partial_{\alpha+1+1_2} \sigma^1(x_o) + (\dots), \end{aligned}$$

where $\underline{\chi}_t = (x^1 + tx^2, x^2, \dots, x^m)$ is the base map of χ_t . Then

$$\begin{aligned} ((J^s \chi_t)_* y_{\alpha-1+1_2}^1) (j_{x_o}^s \sigma) &= y_{\alpha-1+1_2}^1 \circ (J^s \chi_t)^{-1} (j_{x_o}^s \sigma) \\ &= (y_{\alpha-1+1_2}^1 + c_1 t y_{\alpha}^1 + \dots) (j_{x_o}^s \sigma). \quad \square \end{aligned}$$

4. Proof of Theorem 2.2. Because of the invariance of C with respect to the $\mathcal{FM}_{m,n}$ -charts, C is determined by the collection of values

$$\langle C(\lambda, g)_{\rho}, v \rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $\lambda \in \mathcal{C}_{\mathbf{R}^m}^{\infty}(J^s(\mathbf{R}^{m,n}), \bigwedge^m T^* \mathbf{R}^m)$, $v \in T_0 \mathbf{R}^n = V_{(0,0)} \mathbf{R}^{m,n}$, $\rho = j_0^{2s}(\sigma) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$ and $g: \mathbf{R}^m \rightarrow \mathbf{R}$. (Here and from now on the phrase “ C is determined by...” means that if C' is another operator in question giving the same collection of values as C , then $C = C'$.)

For any element $\rho = j_0^{2s}(\sigma) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$, there exists an $\mathcal{FM}_{m,n}$ -map $\nu: \mathbf{R}^{m,n} \rightarrow \mathbf{R}^{m,n}$ transforming $j_0^{2s}(\sigma)$ into $\theta := j_0^{2s}(0) \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$. (Indeed, we can choose $\nu := (x, y - \sigma(x))$, where $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$.) So, we can additionally assume that $\rho = \theta$.

Because of the regularity of C , we can assume that $d_0 g \neq 0$. Then using the invariance of C with respect to a $(0,0)$ -preserving $\mathcal{FM}_{m,n}$ -map of the form $(\varphi(x), y^1, \dots, y^n)$ (it preserves θ), we may additionally assume that $g = x^m + c$, where c is an arbitrary real number.

We can obviously assume that $v \neq 0$. Then using the invariance of C with respect to a respective $\mathcal{FM}_{m,n}$ -map of the form $\text{id}_{\mathbf{R}^m} \times \phi$ for a linear isomorphism $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$, we can additionally assume that $v = \frac{\partial}{\partial y^1} |_{(0,0)}$ (because ϕ preserves θ and g).

Further, we can write $\lambda = L((x^i), (y_{\alpha}^j)) dx^{\mu} + f(x^1, \dots, x^m) dx^{\mu}$, where L and f are arbitrary real valued maps with $L((x^i), (0)) = 0$. By the regularity of C , we can assume that $f(0) \neq 0$. Then, using the invariance of C with respect to $\mathcal{FM}_{m,n}$ -map $b = (F(x^1, \dots, x^m), x^2, \dots, x^m, y^1, \dots, y^n)^{-1}$, where $\frac{\partial}{\partial x^1} F = f$ and $F(0, x^2, \dots, x^m) = 0$, we may additionally assume that $f = 1$ because b preserves θ , g (as $m \geq 2$) and $\frac{\partial}{\partial y^1} |_{(0,0)}$ and it sends dx^{μ} into $f dx^{\mu}$. Consequently, we can write $\lambda = L((x^i), (y_{\alpha}^j)) dx^{\mu} + dx^{\mu}$, where L is an arbitrary real valued map with $L((x^i), (0)) = 0$.

Further, because of the π_s^{2s} -locality of C , using the main result of [5], we may additionally assume that L is a arbitrary polynomial in $((x^i), (y_{\alpha}^j))$ of degree $\leq q$, where q is an arbitrary positive integer.

Further, by the invariance of C with respect to $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1}y^1, \dots, \frac{1}{\tau^n}y^n)$ being $\mathcal{FM}_{m,n}$ -map for any $(\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$, we get the homogeneity condition

$$\begin{aligned} & \left\langle C(L((x^i), (\tau^j y_\alpha^j)) dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= \tau^1 \left\langle C(L((x^i), (y_\alpha^j)) dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle, \end{aligned}$$

see Lemma 3.1 (i). Then by the homogeneous function theorem ([3]), we conclude that $\left\langle C(Ldx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle$ depends linearly on L and C is determined by the collection of values

$$\left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle$$

for $\alpha, \beta \in \mathbf{N}^m$ with $|\beta| \leq q$ and $|\alpha| \leq s$.

Further, by the invariance of C with respect to $\varphi_t^i = (x^1, \dots, \frac{1}{t}x^i, \dots, x^m, y^1, \dots, y^n)$ being $\mathcal{FM}_{m,n}$ -map for any $t \in \mathbf{R} \setminus \{0\}$ and any $i = 1, \dots, m$ and using the fact that $\left\langle C(Ldx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle$ depends linearly on L , we get the condition

$$\begin{aligned} & t^{\beta_i - \alpha_i} \left\langle C(x^\beta y_\alpha^1 dx^\mu + t dx^\mu, t^{\delta_{im}} x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= \left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \end{aligned}$$

because φ_t^i preserves C , θ and $\frac{\partial}{\partial y^1} \Big|_{(0,0)}$ and it sends x^β into $t^{\beta_i} x^\beta$, x^m into $t^{\delta_{im}} x^m$ (the Kronecker delta), y_α^1 into $t^{-\alpha_i} y_\alpha^1$ and dx^μ into $t dx^\mu$, see Lemma 3.1 (ii). Then putting $t \rightarrow 0$, we get the condition

$$\left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle = 0$$

for any $\alpha, \beta \in \mathbf{N}^m$ with both $|\alpha| \leq s$ and $\beta_i > \alpha_i$ for some $i = 1, \dots, m$. Consequently, C is determined by the collection of values

$$\left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \leq s$ and $\beta_1 \leq \alpha_1, \dots, \beta_m \leq \alpha_m$.

Consider $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \leq s$ and $\beta_1 \leq \alpha_1, \dots, \beta_m \leq \alpha_m$. Assume that $\beta \neq (0)$. For example, let $\beta_i \neq 0$ for some $i = 1, \dots, m$. Using the invariance of

C with respect to $\psi^{(i)} = (x^1, \dots, x^m, y^1 + x^i y^1, y^2, \dots, y^n)^{-1}$ (being $\mathcal{FM}_{m,n}$ -map defined over some neighborhood of $0 \in \mathbf{R}^m$), we get

$$\begin{aligned} & \left\langle C(x^{\beta-1_i}(y_\alpha^1 + x^i y_\alpha^1 + y_{\alpha-1_i}^1) dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= \left\langle C(x^{\beta-1_i} y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \end{aligned}$$

because $\psi^{(i)}$ preserves C , $x^{\beta-1_i}$, θ , $\frac{\partial}{\partial y^1} \Big|_{(0,0)}$, dx^μ and $x^m + c$ and it sends y_α^1 into $y_\alpha^1 + x^i y_\alpha^1 + y_{\alpha-1_i}^1$, see Lemma 3.1 (iii). Then

$$\begin{aligned} & \left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= - \left\langle C(x^{\beta-1_i} y_{\alpha-1_i}^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \end{aligned}$$

because $\left\langle C(Ldx^\mu + dx^\mu)_\theta, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle$ depends linearly on L . Repeating this process, we get

$$\begin{aligned} & \left\langle C(x^\beta y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= (-1)^{|\beta|} \left\langle C(y_{(\alpha-\beta)}^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle. \end{aligned}$$

Consequently, C is determined by the collection of values

$$\left\langle C(y_\alpha^1 dx^\mu + dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and all $\alpha \in \mathbf{N}^m$ with $|\alpha| \leq s$.

Let $\alpha \in \mathbf{N}^m$, where $|\alpha| \leq s$, and assume that $\alpha_i \neq 0$ for some $i = 1, \dots, m-1$. For example, let $\alpha_1 \neq 0$. For any $t \in \mathbf{R}$, the $\mathcal{FM}_{m,n}$ -map $\chi_t = (x^1 + tx^2, x^2, \dots, x^m, y^1, \dots, y^n)$ (defined if $m \geq 2$) preserves dx^μ , θ , $\frac{\partial}{\partial y^1} \Big|_{(0,0)}$ and $x^m + c$ and it sends $y_{\alpha-1_1+1_2}^1$ into $y_{\alpha-1_1+1_2}^1 + c_1 t y_\alpha^1 + \dots + c_{\alpha_2+1} t^{\alpha_2+1} y_{(\alpha_1+\alpha_2, 0, \alpha_3, \dots, \alpha_m)}^1$ for some $c_1, \dots \in \mathbf{R}$ with $c_1 \neq 0$, see Lemma 3.1 (iv).

Then using the invariance of C with respect to χ_t , we get

$$\begin{aligned} & \left\langle C((y_{\alpha-1_1+1_2}^1 + c_1 t y_{\alpha}^1 + \dots) dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= \left\langle C(y_{\alpha-1_1+1_2}^1 dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \end{aligned}$$

for any $t \in \mathbf{R}$. Then since $\left\langle C(L dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle$ depends linearly on L , we get $\left\langle C(y_{\alpha}^1 dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle = 0$.

So, C is determined by the collection of values

$$\left\langle C(y_{(0,\dots,0,k)}^1 dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and $k = 0, 1, \dots, r$.

Consequently, C is determined by the collection of (smooth because C is regular) maps $C^{(k)} : \mathbf{R} \rightarrow \mathbf{R}$ for $k = 0, \dots, s$ defined by

$$C^{(k)}(c) dx_{|0}^{\mu} := \left\langle C(y_{(0,\dots,0,k)}^1 dx^{\mu} + dx^{\mu}, x^m + c)_{\theta}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle, \quad c \in \mathbf{R}.$$

More precisely, if C' is another operator in question such that $C^{(k)} = (C')^{(k)}$ for $k = 0, \dots, s$, then $C = C'$.

On the other hand, given a collection of maps $h_l : \mathbf{R} \rightarrow \mathbf{R}$ for $l = 0, \dots, s$, we have $(\sum_{l=0}^s h_l \cdot E^{(l)})^{(k)} = h_k$ for $k = 0, 1, \dots, s$. Indeed, using the coordinate expression of the Euler map $E(\lambda)$ from [3], we have

$$\begin{aligned} & \left(\sum_{l=0}^s h_l \cdot E^{(l)} \right)^{(k)}(c) dx_{|0}^{\mu} \\ &= \sum_{l=0}^s h_l(c) \left\langle E((-1)^l (x^m)^l (y_{(0,\dots,0,k)}^1 dx^{\mu} + dx^{\mu})) \Big|_{j_0^{2s}(0)}, \frac{\partial}{\partial y^1} \Big|_{(0,0)} \right\rangle \\ &= \sum_{l=1}^s h_l(c) (-1)^l (-1)^k \frac{1}{k!} \frac{\partial^k}{\partial (x^m)^k} \frac{\partial}{\partial y_{(0,\dots,0,k)}^k} ((x^m)^l y_{(0,\dots,0,k)}^1) \Big|_{j_0^{2s}(0)} dx_{|0}^{\mu} \\ &= h_k(c) dx_{|0}^{\mu}. \end{aligned}$$

The proof of the theorem is complete.

5. Schema of the proof of Theorem 2.6. The proof of Theorem 2.6 is the following modification of the one of Theorem 2.2.

Because of the invariance of C with respect to the $\mathcal{FM}_{m,n}$ -charts, C is determined by the collection of values

$$\langle C(\lambda, g)_\rho, v \rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $\lambda \in \mathcal{C}_{J^s \mathbf{R}^{m,n}}^\infty(J^s \mathbf{R}^{m,n}, V^* J^s \mathbf{R}^{m,n} \otimes \bigwedge^m T^* \mathbf{R}^m)$, $g \in \mathcal{C}^\infty(\mathbf{R}^m, \mathbf{R})$, $v \in T_0 \mathbf{R}^n = V_{(0,0)} \mathbf{R}^{m,n}$ and $\rho \in J_0^{2s}(\mathbf{R}^m, \mathbf{R}^n) = J_0^{2s}(\mathbf{R}^{m,n})$. Quite similarly as in the proof of Theorem 2.2, we can assume that $\rho = \theta = j_0^{2s}(0)$, $g = x^m + c$ and $v = \frac{\partial}{\partial y^1}|_{(0,0)}$.

Further, we can write $\lambda = \sum L_k^\beta((x^i), (y_\alpha^j)) \tilde{d}y_\beta^k \otimes dx^\mu$, where L_k^β are real valued maps for $k=1, \dots, n$ and all $\beta \in \mathbf{N}^m$ with $|\beta| \leq s$ and where $\tilde{d}h$ denotes the restriction to $VJ^s Y$ of the differential dh of $h: J^s Y \rightarrow \mathbf{R}$.

Because of the π_s^{2s} -locality of C , we may assume that L_k^β are polynomials in $((x^i), (y_\alpha^j))$ of degree $\leq q$, where q is an arbitrary positive integer.

Further, quite similarly as in the proof of Theorem 2.2, by the invariance of C with respect to $\psi_\tau = (x^1, \dots, x^m, \frac{1}{\tau^1} y^1, \dots, \frac{1}{\tau^n} y^n)$ for any $(\tau^1, \dots, \tau^n) \in (\mathbf{R} \setminus \{0\})^n$ and the homogeneous function theorem, we can conclude that $\left\langle C(\lambda, x^m + c)_\theta, \frac{\partial}{\partial y^1}|_{(0,0)} \right\rangle$ is linear in λ and C is determined by the collection of values

$$\left\langle C(x^\beta \tilde{d}y_\alpha^1 \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}|_{(0,0)} \right\rangle$$

for all $\alpha, \beta \in \mathbf{N}^m$ with $|\alpha| \leq s$.

Then by the respective part of proof of Theorem 2.2 with $x^\beta \tilde{d}y_\alpha^1 \otimes dx^\mu$ instead of $x^\beta y_\alpha^1 dx^\mu + dx^\mu$ and using $\tilde{d}(x^i y_\alpha^1) = x^i \tilde{d}y_\alpha^1 \alpha$ (being the consequence of $dh = 0$ on $VJ^s Y$ for any $h: M \rightarrow \mathbf{R}$), we finally conclude that C is determined by the collection of values

$$\left\langle C(\tilde{d}y_{(0,\dots,0,k)}^1 \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}|_{(0,0)} \right\rangle \in \bigwedge^m T_0^* \mathbf{R}^m$$

for all $c \in \mathbf{R}$ and $k = 0, 1, \dots, s$.

Consequently, C is determined by the collection of (smooth) maps $C^{(k)}: \mathbf{R} \rightarrow \mathbf{R}$ for $k = 0, \dots, s$ defined by

$$C^{(k)}(c) dx_\mu^1 := \left\langle C(\tilde{d}y_{(0,\dots,0,k)}^1 \otimes dx^\mu, x^m + c)_\theta, \frac{\partial}{\partial y^1}|_{(0,0)} \right\rangle, \quad c \in \mathbf{R}.$$

On the other hand, given a collection of maps $h_l: \mathbf{R} \rightarrow \mathbf{R}$ for $l = 0, \dots, s$, one can see that $(\sum_{l=0}^s h_l \cdot \mathbf{E}^{(l)})^{(k)} = h_k$ for $k = 0, 1, \dots, s$.

The proof of Theorem 2.6 is complete.

6. Final observations. Let $\mathcal{AB}_{m,n}$ denote the category of all affine bundles $A \rightarrow M$ with m -dimensional bases and n -dimensional fibres and their affine bundle isomorphisms onto open images. It is easy to see that in fact we have also deduced the following results:

Theorem 6.1. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π_s^{2s} -local and $\mathcal{AB}_{m,n}$ -natural (i.e. invariant with respect to $\mathcal{AB}_{m,n}$ -maps) operator*

$$C : \mathcal{C}_M^\infty \left(J^s A, \bigwedge^m T^* M \right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_A^\infty \left(J^{2s} A, V^* A \otimes \bigwedge^m T^* M \right)$$

is $C = \sum_{l=0}^s h_l \cdot E^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \rightarrow \mathbf{R}$, $l = 0, \dots, s$. In other words, the space of all C in question is the free $(s+1)$ -dimensional $\mathcal{C}^\infty(\mathbf{R})$ -module and the operators $E^{(l)}$ for $l = 0, 1, \dots, s$ form the basis in this module.

Theorem 6.2. *Let m, n, s be positive integers. If $m \geq 2$, then any regular, π_s^{2s} -local and $\mathcal{AB}_{m,n}$ -natural operator*

$$C : \mathcal{C}_{J^s A}^\infty \left(J^s A, V^* J^s A \otimes \bigwedge^m T^* M \right) \times \mathcal{C}^\infty(M, \mathbf{R}) \rightarrow \mathcal{C}_A^\infty \left(J^{2s} A, V^* A \otimes \bigwedge^m T^* M \right)$$

is $C = \sum_{l=0}^s h_l \cdot \mathbf{E}^{(l)}$ for some (uniquely determined by C) maps $h_l : \mathbf{R} \rightarrow \mathbf{R}$, $l = 0, \dots, s$. So, the space of all C in question is also the free $(s+1)$ -dimensional $\mathcal{C}^\infty(\mathbf{R})$ -module and the operators $\mathbf{E}^{(l)}$ for $l = 0, 1, \dots, s$ form the basis in this module.

Proof. Indeed, all $\mathcal{FM}_{m,n}$ -maps we used in the previous sections are in fact $\mathcal{AB}_{m,n}$ -maps, except $\mathcal{FM}_{m,n}$ -charts. But they may be replaced by $\mathcal{AB}_{m,n}$ -charts if we study $\mathcal{AB}_{m,n}$ -natural operators. \square

Theorem 2.2 is not true in the vector bundle situation instead of the fibered manifold one because we have:

Example 6.3. Let $\lambda : J^s H \rightarrow \bigwedge^m T^* M$ be an s th order Lagrangian on a vector bundle $H \rightarrow M$. Then the derivative $\mathcal{E}\lambda : J^s H \rightarrow \bigwedge^m T^* M$ of λ with respect to \mathcal{E} , where \mathcal{E} is the Euler (dilatation) vector field on the vector bundle $J^r H \rightarrow M$, is also an s th order Lagrangian on $H \rightarrow M$. Let $U(\lambda) := E(\mathcal{E}\lambda) : J^{2s} H \rightarrow V^* H \otimes \bigwedge^m T^* M$, where E is the Euler operator. Then we have the resulting operator

$$U : \mathcal{C}_M^\infty \left(J^s H, \bigwedge^m T^* M \right) \rightarrow \mathcal{C}_H^\infty \left(J^{2s} H, V^* H \otimes \bigwedge^m T^* M \right).$$

Of course, U is $\mathcal{VB}_{m,n}$ -natural, regular, π_s^{2s} -local and (even) linear but it is not of the form cE , where $\mathcal{VB}_{m,n}$ denotes the category of vector bundles with m -dimensional bases and n -dimensional fibres and their vector bundle

isomorphisms onto open images. Suppose, $U = \sum_{l=0}^s h_l \cdot E^{(l)}$. Then $U(\lambda) = U(\lambda, 1) = h_0(1)E(\lambda)$, i.e. $U = h_0(1)E$ which is a contradiction.

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