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New characterizations of $\mathcal{N}(p, q, s)$ spaces on the unit ball of $\mathbb{C}^n$

Abstract. In this note we provide Holland–Walsh-type characterizations for functions on the $\mathcal{N}(p, q, s)$ spaces on the unit ball for specific values of $p \geq 1$. Characterizations for the holomorphic function spaces $\mathcal{N}(p, q, s)$ were studied extensively by B. Hu and S. Li.

1. Introduction and notation. Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we write $|z|^2 = |z_1|^2 + \cdots + |z_n|^2 = \langle z, z \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product that induces the Euclidean norm in $\mathbb{C}^n$. Let $\mathbb{B}$ denote the open unit ball in $\mathbb{C}^n$, that is $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$. The class of holomorphic functions on the unit ball will be denoted by Hol($\mathbb{B}$). We denote by $dV(z)$ the usual volume measure normalized over the unit ball and set

$$dV_\alpha(z) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}(1 - |z|^2)^\alpha dV(z), \ \alpha \geq -1$$

where $\Gamma(\cdot)$ is the standard Gamma function. Throughout the whole paper by $\Phi_a(z)$ we denote the involutive automorphisms of the unit ball. We have an explicit representation of such functions for $a \in \mathbb{B} \setminus \{0\}$, given by the following formula:

$$\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}, \ z \in \mathbb{B},$$

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where $s_a = \sqrt{1-|a|^2}P_a$, $P_a = \frac{\langle z,a\rangle}{|a|^2}$ is the orthogonal projection of z onto the space spanned by a and $Q_a = I - P_a$. For more details the reader can consult [8].

Now we define the Möbius invariant measure by

$$d\lambda(z) = (1-|z|^2)^{-n-1}dV(z).$$

It is called Möbius invariant because of the following equality:

$$\int_{\mathbb{B}} f(z)d\lambda(z) = \int_{\mathbb{B}} f \circ \Phi_a(z)d\lambda(z).$$

Also, by ∇f we denote the complex gradient of a holomorphic function f, that is

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

and by $\widetilde{\nabla} f$ the Möbius invariant gradient, that is

$$\widetilde{\nabla} f(z) = \nabla(f \circ \Phi_z)(0).$$

Furthermore, by $\mathcal{R}f(z)$ we denote the radial derivative of f, that is

$$\mathcal{R}f(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}.$$

Lastly, whenever we encounter the notation $a \asymp b$, we simply mean that there exist two positive constants C_1, C_2 such that $C_1 a \leq b \leq C_2 a$.

2. Definitions and tools. In this section we give the definition of the $\mathcal{N}(p,q,s)$ spaces and some known characterizations. These will constitute our main tools for proving our results.

Definition 1. Let $f \in \mathrm{Hol}(\mathbb{B})$. For $p \geq 1$, $q > 0$, $s > 0$ we define the holomorphic function spaces $\mathcal{N}(p,q,s)$, or $\mathcal{N}^p_{q,s}$ by:

$$\mathcal{N}^p_{q,s} = \left\{ f \in \mathrm{Hol}(\mathbb{B}) : \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |f(z)|^p (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty \right\}.$$

For different values of the parameters p, q, s we obtain different function spaces. The reader can check [2] for such examples and more details about the $\mathcal{N}(p,q,s)$ spaces. For instance, if we could allow $\mathcal{N}(2,0,s)$, $\frac{n-1}{n} < s < 1$ we would receive the classical Q_s spaces and if the parameter s satisfied $1 < s < \frac{n}{n-1}$, then we would receive the Bloch space $\mathcal{B}$ on the unit ball. By definition, we are not allowed to do so, and this shows that the theory of Q_s spaces is completely independent of the one of $\mathcal{N}(p,q,s)$ spaces. We recall some basic information about the mentioned function spaces. It is already known that for $p \geq 1$, $q, s > 0$ the set of polynomials is dense in $\mathcal{N}(p,q,s)$, if and only if $ns + q > n$. Also for $p \geq 1$, $q, s > 0$ we know that these spaces are functional Banach spaces. For the purpose of our work, we will now mention some known characterizations.

Theorem 2.1. *Let $f \in \mathrm{Hol}(\mathbb{B})$, $p \geq 1$, $q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$. Then $f \in \mathcal{N}(p,q,s)$ if and only if any of the following conditions holds:*

$$I_1 = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\nabla f(z)|^p (1-|z|^2)^{p+q}(1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty,$$

$$I_2 = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\widetilde{\nabla} f(z)|^p (1-|z|^2)^{q}(1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty,$$

$$I_3 = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\mathcal{R} f(z)|^p (1-|z|^2)^{p+q}(1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty.$$

Proof. The proofs for all of the 3 characterizations can be found in [3]. □

The next theorem that we will state is of particular interest for us.

Theorem 2.2. *Suppose $f \in \mathrm{Hol}(\mathbb{B})$, $p \geq 1$, $q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$, $\alpha > q + ns - n - 1$. Then $f \in \mathcal{N}(p,q,s)$ if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(z) dV_\alpha(w) < +\infty$$

Proof. See Theorem 5.12 of [3]. □

3. Motivation and statement of results. Historically, the problem started from [1] and the Holland–Walsh characterization of the classical and well-studied Bloch space $\mathcal{B}$ on the unit disc. Later, in [9], Stroethoff, in a much simpler and elegant way, gave the Holland–Walsh characterization for the Bloch space and also provided a similar characterization for the Besov spaces B_p on the unit disc $\mathbb{D}$ for $p \geq 2$. After defining and studying extensively the Hardy, Bergman, Besov and Dirichlet-type spaces on the unit ball of $\mathbb{C}^n$, some similar double integral characterizations emerged for the cases of Dirichlet, Bergman, Besov, and Bloch-type spaces, and this was followed up by similar characterizations for the Q_s spaces. Later, the authors of [3] treated partially the case of $\mathcal{N}(p,q,s)$ spaces. In the present note, we provide three characterizations that have not been found in [3]. They are similar to the ones presented in [4, 6, 7] for specific values of the parameter p. Let us now state the main results.

Theorem 3.1. *Let $f \in \mathrm{Hol}(\mathbb{B}), q > 0$. Suppose also that the parameters p, q, s satisfy: $\alpha > q + ns - n - 1$, $s > \max\{0, 1 - \frac{q}{n}\}$ and $p \geq 2(n+1+\alpha)$. Then $f \in \mathcal{N}(p,q,s)$, if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|z-w|^{2(n+1+\alpha)}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(w) dV_\alpha(z) < +\infty.$$

Remark 1. Actually, because of the fact that $\alpha > q + ns - n - 1$ the theorem can hold for $p > 2(q + ns)$.

Our second theorem is quite similar to the first one.

Theorem 3.2. *Let $f \in \mathrm{Hol}(\mathbb{B})$, $q > 0$. Suppose also that the parameters p, q, s satisfy: $\alpha > q + ns - n - 1$, $s > \max\{0, 1 - \frac{q}{n}\}$ and $p \geq 2(n + 1 + \alpha)$. Then $f \in \mathcal{N}(p, q, s)$ if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z) - f(w)|^p}{|w - P_w z - s_w Q_w z|^{2(n+1+\alpha)}} \times (1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} dV_\alpha(w) dV_\alpha(z) < +\infty.$$

Our third result is straightforward and can be deduced very easily from Theorem 2.3. The motive behind it can be found in [4] again.

Definition 2. Let $f \in \mathrm{Hol}(\mathbb{B})$. We define the p-Mean Oscillation of f as follows:

$$MO_p(f)(z) = \left(\int_{\mathbb{B}} |f(z) - f(w)|^p \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w\rangle|^{2(n+1)}} dV(w) \right)^{1/p}.$$

By the previous definition, we get:

Theorem 3.3. *Let $f \in \mathrm{Hol}(\mathbb{B})$. If $p \geq 1$, $q > 0$ and $s > \max\{0, 1 - \frac{q}{n}\}$, then $f \in N(p, q, s)$ if and only if*

$$J(f) = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} MO_p^p(f)(z)(1 - |z|^2)^q (1 - |\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty.$$

4. Some lemmata and proofs. Initially, we provide the reader with some lemmata that will constitute our main tools for the proof of Theorem 3.1. Let $\Phi_z(w)$, where $z, w \in \mathbb{B}$, $z \neq w$, be an involutive automorphism of the unit ball on $\mathbb{C}^n$. As in [3], [4] and [5], we consider the *Bergman pseudometric* on the unit ball as:

$$d(z, w) = |\Phi_z(w)|.$$

By

$$E(z, r) = \{w \in \mathbb{B} : |\Phi_z(w)| < r\}$$

we denote the *Bergman-metric ball*, centered at z with radius $r > 0$. The following well-known lemma holds.

Lemma 4.1. *For any $r > 0$ and $z \in \mathbb{B}$, let $E(z, r)$ be the Bergman-metric ball centered at z. Then*

$$(1 - |z|^2) \asymp (1 - |w|^2) \asymp |1 - \langle z, w\rangle|$$

for all $z \in \mathbb{B}$, $w \in E(z, r)$.

Proof. The proof can be found in [10]. □

Lemma 4.2. *Let $f \in \mathrm{Hol}(\mathbb{B})$. If $f(0) = 0$, then for all $p \geq 2(n + 1 + \alpha)$, $\alpha \geq -1$, there exists a positive constant $C > 0$ such that*

$$\int_{\mathbb{B}} \frac{|f(w)|^p}{|w|^{2(n+1+a)}} dV_a(w) \leq C \int_{\mathbb{B}} |f(w)|^p dV_\alpha(w).$$

Proof. The lemma holds for $p = 2(n+1+\alpha)$ (see Lemma 2.2. of [4]). So for all $w \in \mathbb{B}$ we have the trivial inequality

$$\frac{1}{|w|^{2(n+1+\alpha)}} \leq \frac{1}{|w|^p}$$

for all $p \geq 2(n+1+\alpha)$. □

Lemma 4.3. *Let $z, w \in \mathbb{B}, z \neq w$. Then, the following inequalities are true:*

$$|z - \Phi_z(w)| \geq \frac{|w|(1-|z|^2)}{|1-\langle z,w\rangle|}, \tag{4.1}$$

$$|z - \Phi_z(w)|^2 \leq \frac{2(1-|z|^2)}{|1-\langle z,w\rangle|}. \tag{4.2}$$

Proof. The proof can be found again in [10]. □

We are ready to proceed with the proof of Theorem 3.1.

Proof of Theorem 3.1. *Sufficiency*: Let $f \in \mathrm{Hol}(\mathbb{B})$ and p, q, s, α be as in the statement of the theorem. Initially, for the convenience of the reader we set

$$I_a(f) = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|z-w|^{2(n+1+\alpha)}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_\alpha(w)dV_\alpha(z).$$

We assume firstly that $\sup_{a\in\mathbb{B}} I_a(f) < +\infty$. We have to prove that $f \in \mathcal{N}(p,q,s)$. For the upcoming calculations, let

$$k_z(w) = \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}, \;\; z, w \in \mathbb{B}.$$

We will estimate $I_a(f)$ from below. To do so, we apply firstly a change of variables $w = \Phi_z(w)$:

$$I_a(f) = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z) - f\circ\Phi_z(w)|^p}{|z-\Phi_z(w)|^{2(n+1+\alpha)}} \times (1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}k_z(w)dV_\alpha(z)dV_\alpha(w).$$

For convenience in the upcoming calculations, set $F_z(w) = f(z) - f\circ\Phi_z(w)$ and $n+1+\alpha = \gamma$. Applying Fubini's Theorem and Lemma 4.3 (inequality (4.2)), we get

$$I_a(f) \geq C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_a(z) \times \left(\int_{\mathbb{B}} \frac{|F_z(w)|^p}{\frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{n+1+\alpha}}} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}dV_\alpha(w)\right).$$

Fix $0<r<1$ and take $E(z,r)\subset\mathbb{B}$ as the domain of the second integral:

$$\begin{aligned}I_a(f)&\geq C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_a(z)\left(\int_{\mathbb{B}}\frac{|F_z(w)|^p}{|1-\langle z,w\rangle|^{\gamma}}dV_\alpha(w)\right)\\&\geq C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_a(z)\left(\int_{E(z,r)}\frac{|F_z(w)|^p}{|1-\langle z,w\rangle|^{\gamma}}dV_\alpha(w)\right)\\&\geq C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_a(z)\left(\int_{E(z,r)}\frac{|F_z(w)|^p}{(1-|z|^2)^{\gamma}}dV_\alpha(w)\right).\end{aligned}$$

At this point, we move $(1-|z|^2)^{-\gamma}$ to the outer integral. It is now obvious that we replace $(1-|z|^2)^{-n-1-a}dV_a(z)$ with $d\lambda(z)$:

$$I_a(f)\geq C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z)\left(\int_{E(z,r)}|F_z(w)|^p dV_\alpha(w)\right).$$

By the topology induced by the Bergman pseudometric, we can find $\rho<r$ and a Euclidean ball $B(z,\rho)\subset E(z,r)$. Applying the plurisubharmonicity property for the function $|F_z(w)|^p$, $p\geq 1$, we obtain

$$\begin{aligned}I_a(f)&\geq C\int_{\mathbb{B}}|F_z(z)|^p(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z)\\&=C\int_{\mathbb{B}}|f(z)-f(0)|^p(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z).\end{aligned}\tag{4.3}$$

Taking the supremum for $a\in\mathbb{B}$ in (4.3), we get

$$+\infty>\sup_{a\in\mathbb{B}}I_a(f)\geq\sup_{a\in\mathbb{B}}\int_{\mathbb{B}}|f(z)|^p(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z)=\|f\|_{\mathcal{N}},$$

where $\|f\|_{\mathcal{N}}$ denotes the pseudo-norm induced by the definition of the $\mathcal{N}(p,q,s)$ spaces.

Necessity: We assume that $f\in\mathcal{N}(p,q,s)$. We have to show that $\sup_{a\in\mathbb{B}}I_a(f)$ is bounded. We recall that

$$F_z(w)=f(z)-f\circ\Phi_z(w),\ z,w\in\mathbb{B}$$

and observe that $F_z(0)=0$. We begin by applying Fubini's Theorem and a change of variables $w=\Phi_z(w)$, as before:

$$\begin{aligned}I_a(f)=&\left(\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_a(z)\right)\\&\times\left(\int_{\mathbb{B}}\frac{|F_z(w)|^p}{|z-\Phi_z(w)|^{2(n+1+\alpha)}}\frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}dV_\alpha(w)\right).\end{aligned}$$

Now we apply the inequality (4.1) and get

$$I_a(f) \le \left(\int_{\mathbb{B}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_a(z)\right) \times \left(\int_{\mathbb{B}} \frac{|F_z(w)|^p}{\frac{|w|^{2(n+1+\alpha)}(1-|z|^2)^{2(n+1+\alpha)}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} dV_\alpha(w)\right).$$

After we simplify the equal terms in the numerator and denominator, we move the term $(1-|z|^2)^{-(n+1+\alpha)}$ to the outer integral and obtain

$$I_a(f) \le \int_{\mathbb{B}} (1-|z|^2)^{q-(n+1+\alpha)} (1-|\Phi_a(z)|^2)^{ns} dV_a(z) \int_{\mathbb{B}} \frac{|F_z(w)|^p}{|w|^{2(n+1+\alpha)}} dV_\alpha(w).$$

Now we apply Lemma 4.2 for the function $F_z(w)$. After another change of variables we obtain

$$\begin{aligned} I_a(f) &\le C \int_{\mathbb{B}} (1-|z|^2)^{q-(n+1+\alpha)} (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(z) \int_{\mathbb{B}} |F_z(w)|^p dV_a(w) \\ &= C \int_{\mathbb{B}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(z) \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1+a)}} dV_a(w) \\ &= C \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(z) dV_\alpha(w), \end{aligned}$$

hence $I_a(f) < +\infty$, by Theorem 2.3. □

We proceed now with the proof of Theorem 3.2.

Proof of Theorem 3.2. *Sufficiency*: Suppose that

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|w-P_w z - s_w Q_w z|^{2(n+1+\alpha)}} \times (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(w) dV_\alpha(z) < +\infty.$$

Then, by the trivial inequality:

$$\frac{1}{|1-\langle z,w\rangle|} \le \frac{1}{|w-P_w z - s_w Q_w z|}$$

and Theorem 2.3 we deduce that $f \in \mathcal{N}(p,q,s)$.

Necessity: Let $f \in \mathcal{N}(p,q,s)$. We will estimate from above the following integral:

$$J_a(f) = \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|w-P_w z - s_w Q_w z|^{2(n+1+\alpha)}} \times (1-|z|^2)^q (1-|\Phi_a(z)|^2)^{ns} dV_\alpha(w) dV_\alpha(z).$$

Initially, we observe that

$$J_a(f) = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|\Phi_z(w)|^{2(n+1+a)}|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \times (1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_\alpha(w)dV_\alpha(z).$$

We apply the change of variables $w=\Phi_z(w)$ and Fubini's Theorem as in the proof of the previous theorem:

$$I_a(f) = \left(\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_\alpha(z)\right) \times \left(\int_{\mathbb{B}} \frac{|f\circ\Phi_z(w)-f\circ\Phi_z(0)|^p k_z(w)}{|w|^{2(n+1+\alpha)}|1-\langle\Phi_z(w),z\rangle|^{2(n+1+\alpha)}}dV_\alpha(w)\right). \tag{4.4}$$

By the properties of the involutive automorphisms of the unit ball, we know that the following identity holds (see e.g. [8]):

$$\frac{1}{|1-\langle\Phi_z(w),z\rangle|^{2(n+1+\alpha)}} = \frac{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}{(1-|z|^2)^{2(n+1+\alpha)}}. \tag{4.5}$$

Applying (4.5) to (4.4), we obtain

$$J_a(f) = \left(\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z)\right) \times \left(\int_{\mathbb{B}} \frac{|f\circ\Phi_z(w)-f\circ\Phi_z(0)|^p}{|w|^{2(n+1+\alpha)}}dV_\alpha(w)\right).$$

Recall, once again, the previous notation $F_z(w)=f\circ\Phi_z(w)-f\circ\Phi_z(0)$. At this stage, we apply Lemma 4.2. Doing so, we obtain a positive constant $C>0$ such that

$$J_a(f) \le C\int_{\mathbb{B}}(1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}d\lambda(z)\left(\int_{\mathbb{B}}|F_z(w)|^p dV_\alpha(w)\right).$$

Repeating the change of variables and applying Theorem 2.3, we obtain

$$\sup_{a\in\mathbb{B}} J_a(f) \le \sup_{a\in\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \times (1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns}dV_\alpha(z)dV_\alpha(w) < +\infty$$

and the second implication follows. □

Proof of Theorem 3.3. Observe that

$$MO_p^p(f) = \int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|z|^2)^{n+1}dV(w).$$

From Theorem 2.3, choosing $\alpha = 0$, we see that $f \in N(p,q,s)$ if and only if

$$J(f) = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}} \times (1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns} dV(z)dV(w) < +\infty.$$

We write the integral in $J(f)$ as an iterated one and observe that $f \in N(p,q,s)$ if and only if

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}}(1-|z|^2)^{n+1} dV(w)\right) \times (1-|z|^2)^q(1-|\Phi_a(z)|^2)^{ns} d\lambda(z) < +\infty$$

which gives us the desired result. □

5. Discussion and questions. In this final section we pose some comments and questions for the interested reader. The motivation behind our question is clear. We proved Theorem 3.1 and Theorem 3.2 for $p \geq 2(n+1+\alpha)$, $\alpha \geq 1$.

Question 1. *Can we obtain Theorem 3.1 for more general values of the parameter $p \geq 1$? What happens for $1 \leq p < 2(n+1+\alpha)$?*

The answer to this question seems to be technical, as in this case we cannot apply Lemma 4.2 as in our proofs.

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