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On implicative BE algebras

ABSTRACT. We consider some generalizations of BCK algebras (RML, BE, aBE, BE** and aBE** algebras). We investigate the property of implicativity for these algebras. We prove that for any implicative BE** algebra the commutativity property is equivalent to the property of antisymmetry and show that implicative aBE** algebras are commutative BCK algebras. We also show that the class of implicative BE** algebras is a variety.

1. Introduction. In 1966, Y. Imai and K. Iséki [3] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [12], which generalizes the concept of implicative algebras (see [1]). Hundred of papers were written on BCK algebras, and the books [11] and [4]. In [10], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. In 2008, A. Walendziak [14] defined commutative BE algebras and proved that they are BCK algebras. A. Iorgulescu [5] introduced new generalizations of BCK algebras (RML, aBE, BE**, aBE** algebras and many others).

In 1978, K. Iséki and S. Tanaka [8] introduced the notion of implicativity in the theory of BCK algebras. The present paper is a continuation of the author's paper [15], where the property of implicativity for various generalizations of BCK algebras was studied. Here we consider RML, BE, aBE, BE** and aBE** algebras and investigate the implicative property for these

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algebras. We prove that for any implicative \mathbf{BE}^{**} algebra the commutativity property is equivalent to the property of antisymmetry and show that implicative \mathbf{aBE}^{**} algebras are commutative \mathbf{BCK} algebras. We also show that the class of implicative \mathbf{BE}^{**} algebras is a variety.

2. Preliminaries. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We consider the following list of properties ([5]) that can be satisfied by \mathcal{A} :

- (An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \implies x = y$,
- (BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,
- (D) $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$,
- (Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (K) $x \rightarrow (y \rightarrow x) = 1$,
- (L) (Last element) $x \rightarrow 1 = 1$,
- (M) $1 \rightarrow x = x$,
- (Re) (Reflexivity) $x \rightarrow x = 1$,
- (*) $y \rightarrow z = 1 \implies (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,
- (**) $y \rightarrow z = 1 \implies (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$,
- (Tr) (Transitivity) $x \rightarrow y = 1 = y \rightarrow z \implies x \rightarrow z = 1$.

Lemma 2.1 ([5], Proposition 2.1). *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following hold:*

- (i) (M) + (BB) imply (Re), (**),
- (ii) (M) + (BB) + (An) imply (Ex),
- (iii) (M) + (**) imply (Tr),
- (iv) (Re) + (Ex) imply (D),
- (v) (Re) + (L) + (Ex) imply (K),
- (vi) (Re) + (Ex) + (Tr) imply (**),
- (vii) (M) + (*) imply (Tr).

Definition 2.2 ([5]).

1. An *RML algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifying (Re), (M), (L).
2. A *BE algebra* is a RML algebra verifying (Ex).
3. An *aBE algebra* is a BE algebra verifying (An).
4. A *BE^{**} algebra* is a BE algebra verifying (**).
5. An *aBE^{**} algebra* is a BE^{**} algebra verifying (An).
6. A *BCK algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifying (An), (BB), (M), (L).

Denote by **RML**, **BE**, **aBE**, **BE^{**}**, **aBE^{**}** and **BCK** the classes of RML, BE, aBE, BE^{**}, aBE^{**} and BCK algebras, respectively. By definition and Lemma 2.1 (i) and (ii), we have

$$\mathbf{BCK} \subset \mathbf{aBE}^{**} \subset \mathbf{aBE} \subset \mathbf{BE} \subset \mathbf{RML} \text{ and } \mathbf{aBE}^{**} \subset \mathbf{BE}^{**} \subset \mathbf{BE}.$$

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if \mathbf{X} and \mathbf{Y} are classes of algebras, then $\mathbf{X} \longrightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.)

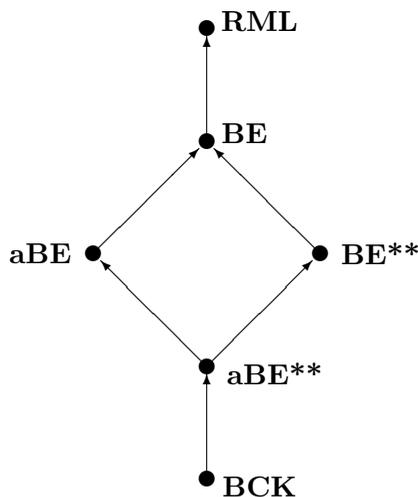


Figure 1

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We define the binary relation \leq by: for all $x, y \in A$,

$$x \leq y \iff x \rightarrow y = 1.$$

It is known that \leq is an order relation in BCK algebras. By definition, in RML and BE algebras, \leq is a reflexive relation; in aBE algebras, \leq is reflexive and antisymmetric. By Lemma 2.1 (iii), in BE^{**} algebras, \leq is reflexive and transitive (i.e., it is a pre-order relation). Lastly, in aBE^{**} algebras, \leq is an order relation.

In [13], S. Tanaka introduced the notion of commutativity in the theory of BCK algebras. A BCK algebra $\mathcal{A} = (A, \rightarrow, 1)$ is called *commutative* if, for all $x, y \in A$,

$$(Com) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

H. Yutani [17] proved that the class of commutative BCK algebras is equationally definable. Commutative BCI and BE algebras were considered in [7] and [14, 2], respectively. K. Iséki [7] proved that any commutative BCI algebra is a BCK algebra. For BE algebras, this was shown in [14]. The property of commutativity for other generalizations of BCK algebras was investigated in [16].

As in the case of BCK algebras, we define:

Definition 2.3. An RML algebra $\mathcal{A} = (A, \rightarrow, 1)$ is called *commutative* if it satisfies (Com).

Lemma 2.4. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then:

- (i) (Com) + (M) imply (An),
- (ii) (Com) + (K) imply (D),
- (iii) (Com) + (Re) + (L) + (Ex) imply (BB).

Proof. The statements (i) and (ii) are proved in [16] (see Proposition 3.3).

(iii) Let \mathcal{A} verify (Com), (Re), (L) and (Ex). By Lemma 2.1 (v), \mathcal{A} verifies (K). Let $x, y, z \in A$. We have

$$\begin{aligned} (y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] \\ &\stackrel{(Ex)}{=} (y \rightarrow z) \rightarrow [y \rightarrow ((z \rightarrow x) \rightarrow x)] \\ &\stackrel{(Com)}{=} (y \rightarrow z) \rightarrow [y \rightarrow ((x \rightarrow z) \rightarrow z)] \\ &\stackrel{(Ex)}{=} (y \rightarrow z) \rightarrow [(x \rightarrow z) \rightarrow (y \rightarrow z)] \stackrel{(K)}{=} 1, \end{aligned}$$

that is, (BB) holds in \mathcal{A} . □

3. Implicative BE algebras. The well-known implicative and positive implicative BCK algebras were introduced by K. Iséki and S. Tanaka [8].

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We first consider the following properties:

- (Im) $(x \rightarrow y) \rightarrow x = x$,
- (pi) $y \rightarrow (y \rightarrow x) = y \rightarrow x$,
- (pimpl) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Remark 3.1. Note that from Theorem 8 of [8] it follows that for BCK algebras, (pimpl) and (pi) are equivalent. By Theorem 9 of [8], in commutative BCK algebras, we have (Im) \iff (pi) \iff (pimpl).

Proposition 3.2. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then

- (i) (Re) + (Im) imply (M),
- (ii) (M) + (Im) imply (L),
- (iii) (Im) implies (pi),
- (iv) (Re) + (M) + (pimpl) + (An) imply (Ex),
- (v) (Re) + (Im) + (BB) + (An) imply (pimpl),
- (vi) (Re) + (pimpl) imply (L),
- (vii) (M) + (K) + (Com) + (pimpl) imply (Im).

Proof. (i)–(iii) follow from Proposition 3.5 of [15].

(iv) follows from Theorem 6.16 of [5].

(v) By above (i)–(iii), \mathcal{A} satisfies (M), (L) and (pi). By Lemma 2.1 (ii) and (v), \mathcal{A} also satisfies (Ex) and (K). Let $x, y, z \in A$. We have

$$\begin{aligned} & (x \rightarrow (y \rightarrow z)) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] \\ & \stackrel{\text{(Ex)}}{=} (x \rightarrow y) \rightarrow [(x \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z)] \\ & \stackrel{\text{(Ex)}}{=} (x \rightarrow y) \rightarrow [(y \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow z)] \\ & \stackrel{\text{(pi)}}{=} (x \rightarrow y) \rightarrow [(y \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow (x \rightarrow z))] \stackrel{\text{(BB)}}{=} 1. \end{aligned}$$

Then

$$(3.1) \quad x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z).$$

On the other hand, from (K) we see that $y \rightarrow (x \rightarrow y) = 1$, and we obtain

$$\begin{aligned} & [(x \rightarrow y) \rightarrow (x \rightarrow z)] \rightarrow (x \rightarrow (y \rightarrow z)) \\ & \stackrel{\text{(Ex)}}{=} [(x \rightarrow y) \rightarrow (x \rightarrow z)] \rightarrow (y \rightarrow (x \rightarrow z)) \\ & \stackrel{\text{(M)}}{=} [y \rightarrow (x \rightarrow y)] \rightarrow [((x \rightarrow y) \rightarrow (x \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z))] \\ & \stackrel{\text{(BB)}}{=} 1. \end{aligned}$$

Hence

$$(3.2) \quad (x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z).$$

Applying (An), from (3.1) and (3.2), we get (pimpl).

(vi) follows from Proposition 6.4 (i) of [5].

(vii) By Lemma 2.4, \mathcal{A} satisfies (An) and (D). Let $x, y, z \in A$. We have

$$\begin{aligned} & ((x \rightarrow y) \rightarrow x) \rightarrow x \stackrel{\text{(Com)}}{=} (x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y) \\ & \stackrel{\text{(pimpl)}}{=} x \rightarrow ((x \rightarrow y) \rightarrow y) \stackrel{\text{(D)}}{=} 1. \end{aligned}$$

Therefore, $(x \rightarrow y) \rightarrow x \leq x$. On the other hand, from (K) we see that $x \leq (x \rightarrow y) \rightarrow x$. Applying (An), we get (Im). \square

Lemma 3.3 ([15], Lemma 3.8). *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then*

$$(\text{Re}) + (\text{Com}) + (\text{Im}) + (\text{BB}) \iff (\text{Re}) + (\text{Com}) + (\text{Im}) + (\text{Ex}).$$

As in the case of BCK algebras, we now define:

Definition 3.4. An RML algebra \mathcal{A} is called *implicative* if it satisfies (Im).

Example 3.5 ([15], Example 3.24). Consider the set $A = \{a, b, c, d, 1\}$ and the operation \rightarrow given by the following table:

\rightarrow	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

We can observe that the properties (Im), (Re), (M) and (L) are satisfied. Hence, $(A, \rightarrow, 1)$ is an implicative RML algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, d)$; (Tr) for $(x, y, z) = (d, c, a)$; (**) for $(x, y, z) = (a, d, c)$.

Example 3.6. Let $A = \{a, b, c, 1\}$ and \rightarrow be defined as follows:

\rightarrow	a	b	c	1
a	1	b	b	1
b	a	1	1	1
c	1	1	1	1
1	a	b	c	1

It is easy to check that the properties (Im), (Re), (M), (L) and (Ex) are satisfied; (An) is not satisfied for $x = b, y = c$; (**) is not satisfied for $x = a, y = b, z = c$. Therefore, $(A, \rightarrow, 1)$ is an implicative BE algebra without (An) and (**).

Example 3.7. Consider the set $A = \{a, b, c, 1\}$ and the operation \rightarrow given by the following table:

\rightarrow	a	b	c	1
a	1	b	c	1
b	a	1	1	1
c	a	1	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \rightarrow, 1)$ satisfies properties (Im), (Re), (M), (L), (Ex), (**). It does not satisfy (An) for $(x, y) = (b, c)$. Hence, \mathcal{A} is an implicative BE** algebra without (An).

Example 3.8 ([15], Example 3.34). Let $A = \{a, b, c, 1\}$ and \rightarrow be defined as follows:

\rightarrow	a	b	c	1
a	1	b	b	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

We can observe that the properties (Im), (An), (Re), (M), (L), (BB) and (Ex) are satisfied. Therefore, $\mathcal{A} = (A, \rightarrow, 1)$ is an implicative BCK algebra.

Denote by **im-RML** the class of all implicative RML algebras. Similarly, if **X** is a subclass of the class **RML**, then **im-X** denotes the class of all implicative algebras belonging to **X**. Examples 3.5–3.8 show that

$$\mathbf{im-BCK} \subseteq \mathbf{im-aBE^{**}} \subset \mathbf{im-BE^{**}} \subset \mathbf{im-BE} \subset \mathbf{im-RML}.$$

Proposition 3.9 ([15], Proposition 3.14). *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra verifying (Re), (D), (***) and (Im). Then*

$$(3.3) \quad y \leq x \implies (x \rightarrow y) \rightarrow y \leq x$$

for all $x, y \in A$.

Theorem 3.10. *If $\mathcal{A} = (A, \rightarrow, 1)$ is an implicative BE^{**} algebra, then \mathcal{A} is commutative if and only if it satisfies (An).*

Proof. Let \mathcal{A} be an implicative BE^{**} algebra. If \mathcal{A} is commutative, then \mathcal{A} satisfies (An) by Lemma 2.4 (i). Conversely, suppose that (An) holds in \mathcal{A} . From Lemma 2.1 (iii)–(v) it follows that \mathcal{A} satisfies (Tr), (D) and (K). By Proposition 3.9, \mathcal{A} satisfies (3.3). Let $x, y \in A$. From (K) we have $x \leq (y \rightarrow x) \rightarrow x$. Applying (***) twice, we obtain

$$(3.4) \quad (x \rightarrow y) \rightarrow y \leq (((y \rightarrow x) \rightarrow x) \rightarrow y) \rightarrow y.$$

By (D), $y \leq (y \rightarrow x) \rightarrow x$, and hence, using (3.3), we get

$$(3.5) \quad (((y \rightarrow x) \rightarrow x) \rightarrow y) \rightarrow y \leq (y \rightarrow x) \rightarrow x.$$

Since \mathcal{A} satisfies (Tr), from inequalities (3.4) and (3.5) we have

$$(x \rightarrow y) \rightarrow y \leq (y \rightarrow x) \rightarrow x.$$

Hence, by (An), we obtain (Com). \square

Proposition 3.11. *In aBE^{**} algebras, we have*

$$(\mathbf{Im}) \iff (\mathbf{Com}) + (\mathbf{pimpl}).$$

Proof. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an aBE^{**} algebra. Assume that \mathcal{A} satisfies (Im). From Theorem 3.10 we conclude that (Com) holds in \mathcal{A} . Applying Lemma 2.4 (iii), we see that \mathcal{A} satisfies (BB). By Proposition 3.2 (v), (pimpl) holds in \mathcal{A} .

Conversely, suppose that \mathcal{A} satisfies (Com) and (pimpl). By Lemma 2.1 (v), (Re) + (L) + (Ex) imply (K). Then, from Proposition 3.2 (vii) we deduce that \mathcal{A} satisfies (Im). \square

Remark 3.12. Since every BCK algebra is an aBE^{**} algebra, from Proposition 3.11 we obtain Theorem 9 of [8].

Corollary 3.13. *Any implicative aBE^{**} algebra is a commutative BCK algebra.*

Proof. Let \mathcal{A} be an implicative aBE** algebra. By Proposition 3.11, \mathcal{A} is commutative. From Lemma 3.3 we see that \mathcal{A} verifies (BB). Hence \mathcal{A} is a BCK algebra. \square

Remark 3.14. Note that from Corollary 3.13 we deduce that **im-BCK** = **im-aBE****.

By Proposition 3.2 (i) and (ii), we obtain:

Proposition 3.15. *An algebra $\mathcal{A} = (A, \rightarrow, 1)$ is an implicative BE algebra if and only if it satisfies the equations (Re), (Ex) and (Im).*

Lemma 3.16. *If $\mathcal{A} = (A, \rightarrow, 1)$ is an implicative BE** algebra, then it satisfies the following condition:*

$$(W) \quad (((y \rightarrow x) \rightarrow x) \rightarrow z) \rightarrow (y \rightarrow z) = 1.$$

Proof. By Lemma 2.1 (iv), \mathcal{A} satisfies (D). Let $x, y, z \in A$. From (D) it follows that $y \leq (y \rightarrow x) \rightarrow x$. Applying (**), we obtain $((y \rightarrow x) \rightarrow x) \rightarrow z \leq y \rightarrow z$. Hence we have (W). \square

The next theorem shows that the class of implicative BE** algebras is a variety.

Theorem 3.17. *An algebra $\mathcal{A} = (A, \rightarrow, 1)$ is an implicative BE** algebra if and only if it satisfies the equations (Re), (Ex), (Im) and (W).*

Proof. If \mathcal{A} is an implicative BE** algebra, then it satisfies (Re), (Ex), (Im) and, by Lemma 3.16, (W).

Conversely, let \mathcal{A} satisfy (Re), (Ex), (Im) and (W). Obviously, \mathcal{A} is an implicative BE algebra. To prove (**), let $x, y, z \in A$ and $y \leq x$. By (M) and (W), $x \rightarrow z \leq y \rightarrow z$, that is, \mathcal{A} satisfies (**). Thus \mathcal{A} is an implicative BE** algebra. \square

Recall the definition of Tarski algebras. A *Tarski algebra* is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ of type (2, 0) satisfying the following axioms ([9]): for all $x, y, z \in A$,

$$(T1) \quad 1 \rightarrow x = x,$$

$$(T2) \quad x \rightarrow x = 1,$$

$$(T3) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

$$(T4) \quad x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z).$$

Note that (T1) is (M), (T2) is (Re), (T3) is (Com) and (T4) is (pimpl).

Theorem 3.18. *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type (2, 0). The following conditions are equivalent:*

- (i) \mathcal{A} is an implicative aBE** algebra,
- (ii) \mathcal{A} satisfies (Re) + (Com) + (Im) + (Ex),
- (iii) \mathcal{A} satisfies (Re) + (Com) + (Im) + (BB),
- (iv) \mathcal{A} is a Tarski algebra.

Proof. (i) implies (ii) and (ii) implies (iii) by Proposition 3.11 and Lemma 3.3, respectively. Now let \mathcal{A} satisfy (Re), (Com), (Im) and (BB). By Proposition 3.2 (i), (Re) + (Im) imply (M), thus, (T1) = (M) holds. By Lemma 2.4 (i), (Com) + (M) imply (An); then, by Proposition 3.2 (v), (Re) + (Im) + (BB) + (An) imply (pimpl); thus, (T4) = (pimpl) holds. Consequently, \mathcal{A} is a Tarski algebra.

Finally, let \mathcal{A} satisfy (M), (Re), (Com) and (pimpl). By Lemma 2.4 (i), (Com) + (M) imply (An); thus, (An) holds. By Proposition 3.2 (vi), (Re) + (pimpl) imply (L); thus, (L) holds. By Proposition 3.2 (iv), (Re) + (M) + (pimpl) + (An) imply (Ex); thus, (Ex) holds.

By Lemma 2.4 (iii), (Com) + (Re) + (L) + (Ex) imply (BB); then, by Lemma 2.1 (i), (M) + (BB) imply (**); thus, (**) holds. By Lemma 2.1 (v), (Re) + (L) + (Ex) imply (K); then, by Proposition 3.2 (vii), (M) + (K) + (Com) + (pimpl) imply (Im); thus (Im) holds.

Consequently, \mathcal{A} is an implicative aBE** algebra, that is, (i) holds. \square

Corollary 3.19. *Let \mathcal{A} be an implicative aBE algebra satisfying (Tr). Then \mathcal{A} is a Tarski algebra.*

Proof. By Lemma 2.1 (vi), \mathcal{A} satisfies (**). Then \mathcal{A} is an implicative aBE** algebra. From Theorem 3.18 we see that \mathcal{A} is a Tarski algebra. \square

Remark 3.20. If \mathcal{A} is an algebra satisfying (M) and (*), then it also satisfies (Tr) by Lemma 2.1 (vii). Therefore, implicative aBE algebras with (*) are Tarski algebras.

Open Problem 3.21. Is there an implicative aBE algebra not satisfying (Tr)?

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