

JAN KUREK and WŁODZIMIERZ M. MIKULSKI

The twisted gauge-natural bilinear brackets on couples of linear vector fields and linear p -forms

ABSTRACT. We completely describe all gauge-natural operators C which send linear $(p+2)$ -forms H on vector bundles E (with sufficiently large dimensional bases) into \mathbf{R} -bilinear operators C_H transforming pairs $(X_1 \oplus \omega_1, X_2 \oplus \omega_2)$ of couples of linear vector fields and linear p -forms on E into couples $C_H(X_1 \oplus \omega_1, X_2 \oplus \omega_2)$ of linear vector fields and linear p -forms on E . Further, we extract all C (as above) such that C_0 is the restriction of the well-known Courant bracket and C_H satisfies the Jacobi identity in Leibniz form for all closed linear $(p+2)$ -forms H .

1. Introduction. All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class C^∞). Maps between manifolds are assumed to be C^∞ .

A vector field X on a vector bundle E is called linear if $\mathcal{L}_L X = 0$, where \mathcal{L} is the Lie derivative and L is the Euler vector field. A p -form ω on a vector bundle E is called linear if $\mathcal{L}_L \omega = \omega$. Let $\Gamma_E^l(T E \oplus \bigwedge^p T^* E)$ denote the space of couples $X \oplus \omega$ of linear vector fields X and linear p -forms ω on E .

2010 *Mathematics Subject Classification.* 53A55, 53A45, 53A99.

Key words and phrases. Natural operator, linear vector field, linear p -form, Jacobi identity in Leibniz form.

Let $\mathcal{VB}_{m,n}$ be the category of n -rank vector bundles with m -dimensional bases and their vector bundle isomorphism onto images. A $\mathcal{VB}_{m,n}$ -gauge-natural operator

$$C : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(T \oplus \bigwedge^p T^* \right) \times \Gamma^l \left(T \oplus \bigwedge^p T^* \right), \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \right)$$

sending linear $(p+2)$ -forms $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$ on $\mathcal{VB}_{m,n}$ -objects E into \mathbf{R} -bilinear operators

$$C_H : \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \rightarrow \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of regular operators (functions)

$$\begin{aligned} C : \Gamma_E^l \left(\bigwedge^{p+2} T^* E \right) \\ \rightarrow \text{Lin}_2 \left(\Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right), \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \right) \end{aligned}$$

for all $\mathcal{VB}_{m,n}$ -objects E , where $\text{Lin}_2(U \times V, W)$ denotes the vector space of all bilinear (over \mathbf{R}) functions $U \times V \rightarrow W$ for any real vector spaces U, V, W .

The first main result of the article is the following theorem.

Theorem 1.1. *Let $m, p \geq 1$ and $n \geq 1$ be fixed integers such that $m \geq p+2$. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(T \oplus \bigwedge^p T^* \right) \times \Gamma^l \left(T \oplus \bigwedge^p T^* \right), \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \right)$$

is of the form

$$\begin{aligned} (1) \quad C_H(\rho^1, \rho^2) = & a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^1 + b_3 di_{X^1} \omega^2 \\ & + b_4 di_{X^2} \omega^1 + b_5 \mathcal{L}_{X^1} di_L \omega^2 + b_6 \mathcal{L}_{X^2} di_L \omega^1 \\ & + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} di_{X^1} H \\ & + c_4 i_L i_{X^1} di_{X^2} H + c_5 i_L di_{X^2} i_{X^1} H\} \end{aligned}$$

for arbitrary (uniquely determined by C) reals $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$, where $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus \bigwedge^p T^* E)$, $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$, and where $[-, -]$ is the usual bracket on vector fields, \mathcal{L} is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

A $\mathcal{VB}_{m,n}$ -gauge-natural operator C as above satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms if

$$(2) \quad C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3))$$

for all closed linear $(p+2)$ -forms $H \in \Gamma_E^l(\wedge^{p+2} T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus \wedge^p T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$ -objects E .

For example, the twisted Dorfman–Courant bracket given by

$$(3) \quad [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_H := [X^1, X^2] \oplus \{\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + i_{X^1}i_{X^2}H\}$$

is a gauge-natural operator in question satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms.

The second main result of the article is the following theorem.

Theorem 1.2. *If additionally $m \geq p + 3$, then any gauge-natural operator C as above satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms and the initial condition $C_0 = [[-, -]]_0$ satisfies the equality*

$$(4) \quad C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH}$$

for any closed linear $(p+2)$ -form $H \in \Gamma_E^l(\wedge^{p+2} T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus \wedge^p T^*E)$, where $[[-, -]]_H$ is the (above) twisted (H -twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Theorems 1.1 and 1.2 for $p = 1$ are proved in [4].

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over \mathbf{R}^m with the standard fibre \mathbf{R}^n and let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. The gauge-natural bilinear brackets on couples of linear vector fields and linear p -forms. Let m, n, p be positive integers.

Let $E = (E \rightarrow M)$ be a vector bundle from $\mathcal{VB}_{m,n}$.

Applying the tangent and the cotangent functors, we obtain double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$.

A vector field X on E is called linear if it is a vector bundle map $X : E \rightarrow TE$ between $E \rightarrow M$ and $TE \rightarrow TM$.

Equivalently, a vector field X on E is linear if it has an expression

$$X = \sum_{i=1}^m a^i(x^1, \dots, x^m) \frac{\partial}{\partial x^i} + \sum_{j,k=1}^n b_j^k(x^1, \dots, x^m) y^j \frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E .

Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X = 0$, where \mathcal{L} denotes the Lie derivative and L is the Euler vector field on E (in vector bundle coordinates $L = \sum_{j=1}^n y^j \frac{\partial}{\partial y^j}$).

Equivalently, a vector field X on E is linear if $(a_t)_* X = X$ for any $t > 0$, where $a_t : E \rightarrow E$ is the fibre-homothety by t .

A p -form ω on E is called linear if the induced vector bundle morphism

$$\omega^\sharp : \oplus^{k-1} TE \rightarrow T^*E$$

over the identity on E is also a vector bundle morphism over a map $\oplus^{k-1}TM \rightarrow E^*$ on the other side of the double vector bundle.

Equivalently, a p -form ω on E is linear if it has an expression

$$\omega = \sum a_{i_1, \dots, i_p, j}(x) y^j dx^{i_1} \wedge \dots \wedge dx^{i_p} + \sum b_{i_1, \dots, i_{p-1}, j}(x) dy^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E .

Equivalently, a p -form ω on E is linear iff $\mathcal{L}_L \omega = \omega$.

Equivalently, a p -form ω on E is linear iff $(a_{\frac{1}{t}})_* \omega = t\omega$ for any $t > 0$.

We have the following definition being a modification of the general one from [1].

Definition 2.1. A $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator

$$A : \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \times \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \rightsquigarrow \Gamma^l \left(T \oplus \bigwedge^p T^* \right)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of \mathbf{R} -bilinear operators

$$A : \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \rightarrow \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right)$$

for all $\mathcal{VB}_{m,n}$ -objects E , where $\Gamma_E^l(TE \oplus \bigwedge^p T^* E)$ is the vector space of linear sections of $TE \oplus \bigwedge^p T^* E$.

Remark 2.2. The $\mathcal{VB}_{m,n}$ -invariance of A means that if

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right)$$

and

$$(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2) \in \Gamma_{\bar{E}}^l \left(T\bar{E} \oplus \bigwedge^p T^*\bar{E} \right) \times \Gamma_{\bar{E}}^l \left(T\bar{E} \oplus \bigwedge^p T^*\bar{E} \right)$$

are φ -related by an $\mathcal{VB}_{m,n}$ -map $\varphi : E \rightarrow \bar{E}$ (i.e., $\bar{X}^i \circ \varphi = T\varphi \circ X^i$ and $\bar{\omega}^i \circ \varphi = \bigwedge^p T^* \varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\bar{X}^1 \oplus \bar{\omega}^1, \bar{X}^2 \oplus \bar{\omega}^2)$.

In [2], we proved the following result.

Theorem 2.3. *Let $m, n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p + 1$. Any $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator*

$$A : \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \times \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \rightsquigarrow \Gamma^l \left(T \oplus \bigwedge^p T^* \right)$$

is of the form

$$(5) \quad A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^1 + b_3 di_{X^1} \omega^2 + b_4 di_{X^2} \omega^1 + b_5 \mathcal{L}_{X^1} di_L \omega^2 + b_6 \mathcal{L}_{X^2} di_L \omega^1\}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$.

3. The twisted gauge-natural bilinear brackets on couples of linear vector fields and linear p -forms.

Definition 3.1. A $\mathcal{VB}_{m,n}$ -gauge-natural operator

$$C : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(T \oplus \bigwedge^p T^* \right) \times \Gamma^l \left(T \oplus \bigwedge^p T^* \right), \Gamma^l \left(T \oplus \bigwedge^p T^* \right) \right)$$

sending linear $(p+2)$ -forms $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$ on $\mathcal{VB}_{m,n}$ -objects E into \mathbf{R} -bilinear operators

$$C_H : \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \rightarrow \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of regular operators (functions)

$$C : \Gamma_E^l \left(\bigwedge^{p+2} T^* E \right) \rightarrow \text{Lin}_2 \left(\Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right), \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \right)$$

for all $\mathcal{VB}_{m,n}$ -objects E , where $\text{Lin}_2(U \times V, W)$ denotes the vector space of all bilinear (over \mathbf{R}) functions $U \times V \rightarrow W$ for any real vector spaces U, V, W .

Remark 3.2. The invariance of C means that if $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$ and $\tilde{H} \in \Gamma_{\tilde{E}}^l(\bigwedge^{p+2} T^* \tilde{E})$ are φ -related and

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right) \times \Gamma_E^l \left(TE \oplus \bigwedge^p T^* E \right)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma_{\tilde{E}}^l \left(T\tilde{E} \oplus \bigwedge^p T^* \tilde{E} \right) \times \Gamma_{\tilde{E}}^l \left(T\tilde{E} \oplus \bigwedge^p T^* \tilde{E} \right)$$

are also φ -related by a $\mathcal{VB}_{m,n}$ -map $\varphi : E \rightarrow \tilde{E}$, then so are $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $C_{\tilde{H}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.

The regularity of C means that C transforms smoothly parametrized families $(H_t, X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$ into smoothly parametrized families $C_{H_t}(X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$.

Example 3.3. The twisted Dorfman–Courant bracket

$$(6) \quad [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_H := [X^1, X^2] \oplus \{ \mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 + i_{X^1} i_{X^2} H \}$$

is a gauge natural operator in the sense of Definition 3.1.

Remark 3.4. Quite similarly, one can introduce the concepts of $\mathcal{VB}_{m,n}$ -gauge-natural operators

$$\begin{aligned} \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T)), \\ \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l \left(\bigwedge^p T^* \right) \right), \\ \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l(T) \right), \\ &\vdots \\ \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l \left(\bigwedge^p T^* \right) \right). \end{aligned}$$

For example, a $\mathcal{VB}_{m,n}$ -gauge-natural operator

$$\Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l(T) \right)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of regular operators (functions)

$$\Gamma_E^l \left(\bigwedge^{p+2} T^* E \right) \rightarrow \text{Lin}_2 \left(\Gamma_E^l(T E) \times \Gamma_E^l \left(\bigwedge^p T^* E \right), \Gamma_E^l(T E) \right)$$

for all $\mathcal{VB}_{m,n}$ -objects E .

Lemma 3.5. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator C in the sense of Definition 3.1 can be considered (in the obvious way) as the system $C = (C^1, C^2, \dots, C^8)$ of $\mathcal{VB}_{m,n}$ -gauge natural operators

$$\begin{aligned} C^1 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T)), \\ C^2 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l \left(\bigwedge^p T^* \right) \right), \\ &\vdots \\ C^8 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) &\rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l \left(\bigwedge^p T^* \right) \right). \end{aligned}$$

Proof. The lemma is obvious. □

We prove the following theorem corresponding to Theorem 1.1.

Theorem 3.6. Let $m, p \geq 1$ and $n \geq 1$ be fixed integers such that $m \geq p+2$. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator C in the sense of Definition 3.1 is of the

form

$$(7) \quad \begin{aligned} C_H(\rho^1, \rho^2) = & a[X^1, X^2] \oplus \{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3di_{X^1}\omega^2 \\ & + b_4di_{X^2}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^2 + b_6\mathcal{L}_{X^2}di_L\omega^1 \\ & + c_1i_{X^1}i_{X^2}H + c_2i_Li_{X^1}i_{X^2}dH + c_3i_Li_{X^2}di_{X^1}H \\ & + c_4i_Li_{X^1}di_{X^2}H + c_5i_Ldi_{X^2}i_{X^1}H\} \end{aligned}$$

for arbitrary (uniquely determined by C) reals $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$, where $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(T E \oplus \bigwedge^p T^* E)$, $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$.

Proof. For $p = 1$, our theorem is the main result of [4]. So, Theorem 3.6 for $p = 1$ holds. So we may assume that $p \geq 2$.

Consider a $\mathcal{VB}_{m,n}$ -gauge-natural operator C in the sense of Definition 3.1. We can easily see that C_0 is a $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator in the sense of Definition 2.1. Hence, replacing C by $C - C_0$ and using Theorem 2.3, we may assume that

$$C_0 = 0.$$

So, because of Lemma 3.5, our theorem is an immediate consequence of Lemmas 3.7–3.14, below. \square

Lemma 3.7. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^1 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow \text{Lin}_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T))$$

such that $C_0^1 = 0$ is 0.

Proof. Using the invariance of C^1 with respect to the fiber homotheties, we get $C_{tH}^1(X, X_1) = C_H^1(X, X_1)$ for any linear vector fields X and X_1 and any linear $(p+2)$ -form H on a $\mathcal{VB}_{m,n}$ -object E and any $t > 0$. Putting $t \rightarrow 0$, we get $C_H^1(X, X_1) = C_0^1(X, X_1)$. Then (by $C_0^1 = 0$) $C_H^1(X, X_1) = 0$. So, $C^1 = 0$. \square

Lemma 3.8. *Let $m, p \geq 2$ and $n \geq 1$ be fixed integers such that $m \geq p+2$. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^2 : \Gamma^l\left(\bigwedge^{p+2} T^*\right) \rightsquigarrow \text{Lin}_2\left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l\left(\bigwedge^p T^*\right)\right)$$

such that $C_0^2 = 0$ is of the form

$$(8) \quad \begin{aligned} C_H^2(X^1, X^2) = & c_1i_{X^1}i_{X^2}H + c_2i_Li_{X^1}i_{X^2}dH + c_3i_Li_{X^2}di_{X^1}H \\ & + c_4i_Li_{X^1}di_{X^2}H + c_5i_Ldi_{X^2}i_{X^1}H \end{aligned}$$

for arbitrary (uniquely determined by C^2) reals c_1, c_2, c_3, c_4, c_5 , where $X^1, X^2 \in \Gamma_E^l(T E)$ and $H \in \Gamma_E^l(\bigwedge^{p+2} T^* E)$.

Proof. Consider arbitrary linear $(p+2)$ -forms H and \tilde{H} and linear vector fields X, \tilde{X}, X_1 and \tilde{X}_1 on $E = \mathbf{R}^{m,n}$.

By the non-linear Peetre theorem (Theorem 19.10 (for $f = 0$) in [1]), there is a positive integer r (independent of $H, \tilde{H}, X, \tilde{X}, X_1, \tilde{X}_1$) such that the conditions

$$j_0^r(H) = j_0^r(\tilde{H}), j_0^r(X) = j_0^r(\tilde{X}), j_0^r(X_1) = j_0^r(\tilde{X}_1) \quad (0 \in \mathbf{R}^m)$$

imply

$$j_0^0(C_{tH}^2(tX, tX_1)) = j_0^0(C_{t\tilde{H}}^2(t\tilde{X}, t\tilde{X}_1)) \quad (0 \in \mathbf{R}^m)$$

for a sufficiently small real number $t > 0$ (depending on $H, \tilde{H}, X, \tilde{X}, X_1, \tilde{X}_1$).

Further, using the invariance of C^2 with respect to the fiber homotheties, we get

$$(9) \quad C_{tH}^2(X, X_1) = tC_H^2(X, X_1)$$

for all $t > 0$. (Then $C_{tH}^2(tX, tX_1) = t^3C_H^2(X, X_1)$ for all $t > 0$.)

Then the conditions

$$j_0^r(H) = j_0^r(\tilde{H}), j_0^r(X) = j_0^r(\tilde{X}), j_0^r(X_1) = j_0^r(\tilde{X}_1)$$

imply

$$C_H^2(X, X_1)|_0 = C_{\tilde{H}}^2(\tilde{X}, \tilde{X}_1)|_0 \quad (0 \in \mathbf{R}^m).$$

Consequently, C^2 is of finite order r . Then $C_H^2(X, X_1)$ is linear in H because of (9) and the homogeneous function theorem.

It is obvious that C^2 is determined by the values

$$(10) \quad i_{X_3} \dots i_{X_{p+2}} C_H^2(X_1, X_2)|_u \in \mathbf{R}$$

for all points $u \in \mathbf{R}_0^{m,n}$, all vectors $X_3, \dots, X_{p+2} \in T_u \mathbf{R}^{m,n}$, all linear vector fields X_1 and X_2 and all linear $(p+2)$ -forms H on $\mathbf{R}^{m,n}$, where i is the insertion derivative.

Using the 3-linearity of C^2 , we can assume that the underlined vector field \underline{X}_2 of X_2 is of the form $\underline{X}_2 = fY$ for some ‘‘constant’’ vector field Y on \mathbf{R}^m and some $f : \mathbf{R}^m \rightarrow \mathbf{R}$. We can also assume that $u \neq 0$ and

$$T\pi \circ X_1|_u, Y|_0, T\pi(X_3), \dots, T\pi(X_{p+2})$$

are linearly independent (here we use $m \geq p+2$), where π is the bundle projection of $E = \mathbf{R}^{m,n}$. Then, using the $\mathcal{VB}_{m,n}$ -invariance of C^2 , the 3-linearity of C^2 and the vector bundle version of the Frobenius theorem, we can write

$$(11) \quad u = e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n = \mathbf{R}_0^{m,n},$$

$$H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}} \text{ or } H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}},$$

$$(12) \quad X_1 = \frac{\partial}{\partial x^1},$$

$$(X_2 = x^\beta \frac{\partial}{\partial x^2} \text{ or } X_2 = x^\beta y^k \frac{\partial}{\partial y^l})$$

and

$$(13) \quad X_3 = \frac{\partial}{\partial x^3|_u}, \dots, X_{p+2} = \frac{\partial}{\partial x^{p+2}|_u},$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ are m -tuples of non-negative integers, i_1, \dots, i_{p+2} are integers with $1 \leq i_1 < i_2 < \dots < i_{p+2} \leq m$, j_1, \dots, j_{p+1} are integers with $1 \leq j_1 < j_2 < \dots < j_{p+1} \leq m$ and k, l are numbers from $\{1, \dots, n\}$. Let us assume additionally that

$$(14) \quad i_{X_3} \dots i_{X_{p+2}} C_H^2(X_1, X_2)|_u \neq 0.$$

First we consider the case $H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}}$ and $X_2 = x^\beta \frac{\partial}{\partial x^2}$. Then using the invariance of C^2 with respect to $(\tau_1 x^1, \dots, \tau_m x^m, y^1, \dots, y^n)$, we get the condition

$$\begin{aligned} & \tau_1 \dots \tau_{p+2} \cdot i_{X_3} \dots i_{X_{p+2}} C_H^2(X_1, X_2)|_u \\ &= \tau^\alpha \cdot \tau^\beta \cdot \tau_{i_1} \dots \tau_{i_{p+2}} \cdot i_{X_3} \dots i_{X_{p+2}} C_H^2(X_1, X_2)|_u. \end{aligned}$$

Then $\alpha = (0)$, $\beta = (0)$, $i_1 = 1$ and ... and $i_{p+2} = p + 2$, i.e.,

$$H = y^k dx^1 \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = \frac{\partial}{\partial x^2}.$$

Next, we consider the case $H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}}$ and $X_2 = x^\beta y^k \frac{\partial}{\partial y^l}$. Then (using similar arguments), we get

$$H = dy^k \wedge dx^1 \wedge dx^3 \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = y^k \frac{\partial}{\partial y^l}.$$

Similarly, in the case $H = x^\alpha y^k dx^{i_1} \wedge \dots \wedge dx^{i_{p+2}}$ and $X_2 = x^\beta y^k \frac{\partial}{\partial y^l}$, we get a contradiction with (14).

Similarly, in the case $H = x^\alpha dy^k \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{p+1}}$ and $X_2 = x^\beta \frac{\partial}{\partial x^2}$, we get

$$(15) \quad (H = x^i dy^k \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = \frac{\partial}{\partial x^2})$$

or

$$(16) \quad (H = dy^k \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{p+2} \text{ and } X_2 = x^i \frac{\partial}{\partial x^2})$$

for some $i = 1, \dots, p + 2$, where \widehat{a} means that a is dropped. If $i = i_o \geq 4$, then using the invariance of C^2 when replacing x^3 by x^{i_o} (and vice-versa), we see that the value (10) for $i = i_o$ is equal (modulo signum) to the value (10) for $i_o = 3$. So, we can assume that $i = 1, 2, 3$.

Consequently, the operator C^2 is determined by the $\mathcal{VB}_{3,n}$ -gauge-natural operator

$$\tilde{C}^2 : \Gamma^l \left(\bigwedge^3 T^* \right) \rightsquigarrow \text{Lin}_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T^*))$$

given by

$$\tilde{C}_H^2(\tilde{X}_1, \tilde{X}_2) := j^* i_{Y_4} \dots i_{Y_{p+2}} C_{\tilde{H} \wedge \omega_o}^2(\tilde{X}_1 \times 0, \tilde{X}_2 \times 0),$$

$\tilde{X}_1, \tilde{X}_2 \in \Gamma_{\tilde{E}}^l(T\tilde{E})$, $\tilde{H} \in \Gamma_{\tilde{E}}^l(\bigwedge^3 T^*\tilde{E})$, where \tilde{E} is a $\mathcal{VB}_{3,n}$ -object with base \tilde{M} , x^4, \dots, x^m are the usual coordinates on \mathbf{R}^{m-3} , $\omega_o := dx^4 \wedge \dots \wedge dx^{p+2}$ (since $p \geq 2$, then $m \geq p+2 \geq 4$, and then ω_o is well defined), $Y_4 := \frac{\partial}{\partial x^4}$ and ... and $Y_{p+2} := \frac{\partial}{\partial x^{p+2}}$ are considered as linear vector fields on the $\mathcal{VB}_{m,n}$ -object $E := \tilde{E} \times \mathbf{R}^{m-3}$ with the base $\tilde{M} \times \mathbf{R}^{m-3}$, $j : \tilde{E} \rightarrow E$ is the inclusion $y \rightarrow (y, 0)$ and j^* denotes the pull-back with respect to j . Of course, $\tilde{C}_0^2 = 0$.

By Theorem 3.6 for $p = 1$ (which is proved in [4]), the vector space of all $\mathcal{VB}_{3,n}$ -gauge-natural operators

$$\tilde{C} : \Gamma^l \left(\bigwedge^3 T^* \right) \rightsquigarrow \text{Lin}_2(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l(T^*))$$

with $\tilde{C}_0 = 0$ is of dimension ≤ 5 . Consequently, the vector space of all $\mathcal{VB}_{m,n}$ -gauge-natural operators

$$C^2 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l \left(\bigwedge^p T^* \right) \right)$$

with $C_0^2 = 0$ is of dimension ≤ 5 .

On the other hand, the system of $\mathcal{VB}_{m,n}$ -gauge-natural operators

$$D^1, D^2, D^3, D^4, D^5 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l(T), \Gamma^l \left(\bigwedge^p T^* \right) \right)$$

defined by

$$\begin{aligned} D_H^1(X^1, X^2) &:= i_{X^1} i_{X^2} H, \\ D_H^2(X^1, X^2) &:= i_L i_{X^1} i_{X^2} dH, \\ D_H^3(X^1, X^2) &:= i_L i_{X^2} di_{X^1} H, \\ D_H^4(X^1, X^2) &:= i_L i_{X^1} di_{X^2} H, \\ D_H^5(X^1, X^2) &:= i_L di_{X^2} i_{X^1} H \end{aligned}$$

is linearly independent. Indeed, if

$$a^1 D^1 + a^2 D^2 + a^3 D^3 + a^4 D^4 + a^5 D^5 = 0,$$

then (in particular)

$$\begin{aligned} a^1 i_{X^1} i_{X^2} H + a^2 i_L i_{X^1} i_{X^2} dH + a^3 i_L i_{X^2} di_{X^1} H \\ + a^4 i_L i_{X^1} di_{X^2} H + a^5 i_L di_{X^2} i_{X^1} H = 0 \end{aligned}$$

for any linear 3-form \tilde{H} and any linear vector fields \tilde{X}^1, \tilde{X}^2 on $\mathbf{R}^{3,n}$, where $H = \tilde{H} \wedge \omega_o \in \Gamma_{\mathbf{R}^{m,n}}^l(\wedge^{p+2} T^* \mathbf{R}^{m,n})$ and $X^1 = \tilde{X}^1 \times 0$, $X^2 = \tilde{X}^2 \times 0 \in \Gamma_{\mathbf{R}^{m,n}}^l(T\mathbf{R}^{m,n})$ and ω_o is as above. Then

$$\begin{aligned} (a^1 i_{\tilde{X}^1} i_{\tilde{X}^2} \tilde{H} + a^2 i_L i_{\tilde{X}^1} i_{\tilde{X}^2} d\tilde{H} + a^3 i_L i_{\tilde{X}^2} di_{\tilde{X}^1} \tilde{H} \\ + a^4 i_L i_{\tilde{X}^1} di_{\tilde{X}^2} \tilde{H} + a^5 i_L di_{\tilde{X}^2} i_{\tilde{X}^1} \tilde{H}) \wedge \omega_o = 0 \end{aligned}$$

for any $\tilde{H}, \tilde{X}^1, \tilde{X}^2$ as above. Then

$$\begin{aligned} a^1 i_{\tilde{X}^1} i_{\tilde{X}^2} \tilde{H} + a^2 i_L i_{\tilde{X}^1} i_{\tilde{X}^2} d\tilde{H} + a^3 i_L i_{\tilde{X}^2} di_{\tilde{X}^1} \tilde{H} \\ + a^4 i_L i_{\tilde{X}^1} di_{\tilde{X}^2} \tilde{H} + a^5 i_L di_{\tilde{X}^2} i_{\tilde{X}^1} \tilde{H} = 0 \end{aligned}$$

for any $\tilde{H}, \tilde{X}^1, \tilde{X}^2$ as above. Then

$$a^1 = a^2 = a^3 = a^4 = a^5 = 0,$$

because the collection of operators D^1, D^2, D^3, D^4, D^5 is linearly independent for $p = 1$ and $m = 3$ and $n \geq 1$, see [4].

So, the dimension argument ends the proof of our lemma. \square

Lemma 3.9. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^3 : \Gamma^l \left(\wedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l \left(\wedge^p T^* \right), \Gamma^l(T) \right)$$

(not necessarily satisfying $C_0^3 = 0$) is 0.

Proof. Using the invariance of C^3 with respect to the fiber homotheties, we get $C_{tH}^3(X, t\omega) = C_H^3(X, \omega)$ for any linear vector field X , any linear p -form ω , any linear $(p+2)$ -form H on a $\mathcal{VB}_{m,n}$ -object E and any $t > 0$. Putting $t \rightarrow 0$, we get $C_H^3(X, \omega) = C_0^3(X, 0) = 0$. So, $C^3 = 0$. \square

Lemma 3.10. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^4 : \Gamma^l \left(\wedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l(T) \times \Gamma^l \left(\wedge^p T^* \right), \Gamma^l \left(\wedge^p T^* \right) \right)$$

such that $C_0^4 = 0$ is 0.

Proof. Using the invariance of C^4 with respect to the fiber homotheties, we get $C_{tH}^4(X, t\omega) = tC_H^4(X, \omega)$ for any linear vector field X , any linear p -form ω , any linear $(p+2)$ -form H on a $\mathcal{VB}_{m,n}$ -object E and any $t > 0$. Then $C_{tH}^4(X, \omega) = C_H^4(X, \omega)$. Putting $t \rightarrow 0$, we get $C_H^4(X, \omega) = C_0^4(X, \omega)$. Then (by the assumption $C_0^4 = 0$), $C_H^4(X, \omega) = 0$. So, $C^4 = 0$. \square

Lemma 3.11. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^5 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l(T), \Gamma^l(T) \right)$$

(not necessarily satisfying $C_0^5 = 0$) is 0.

Proof. It is sufficient to apply Lemma 3.9 for $C_H^3(X, \omega) := C_H^5(\omega, X)$. \square

Lemma 3.12. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^6 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l(T), \Gamma^l \left(\bigwedge^p T^* \right) \right)$$

such that $C_0^6 = 0$ is 0.

Proof. It is sufficient to apply Lemma 3.10 for $C_H^4(X, \omega) := C_H^6(\omega, X)$. \square

Lemma 3.13. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^7 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l(T) \right)$$

(not necessarily satisfying $C_0^7 = 0$) is 0.

Proof. Using the invariance of C^7 with respect to the fiber homotheties, we get $C_{tH}^7(t\omega, t\omega^1) = C_H^3(\omega, \omega^1)$ for any linear p -forms ω and ω^1 , any linear $(p+2)$ -form H on a $\mathcal{VB}_{m,n}$ -object E and any $t > 0$. Putting $t \rightarrow 0$, we get $C_H^7(\omega, \omega^1) = C_0^7(0, 0) = 0$. So, $C^7 = 0$. \square

Lemma 3.14. *Let m, n, p be fixed positive integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator*

$$C^8 : \Gamma^l \left(\bigwedge^{p+2} T^* \right) \rightsquigarrow \text{Lin}_2 \left(\Gamma^l \left(\bigwedge^p T^* \right) \times \Gamma^l \left(\bigwedge^p T^* \right), \Gamma^l \left(\bigwedge^p T^* \right) \right)$$

(not necessarily satisfying $C_0^8 = 0$) is 0.

Proof. Using the invariance of C^8 with respect to the fiber homotheties, we get $C_{tH}^8(t\omega, t\omega_1) = tC_H^8(\omega, \omega_1)$ for any linear p -forms ω and ω_1 , any linear $(p+2)$ -form H on a $\mathcal{VB}_{m,n}$ -object E and any $t > 0$. Then $C_{tH}^8(\omega, t\omega_1) = C_H^8(\omega, \omega_1)$. Putting $t \rightarrow 0$, we get $C_H^8(\omega, \omega_1) = C_0^8(\omega, 0) = 0$. So, $C^8 = 0$. \square

4. The generalized twisted D-C brackets with the Jacobi identity in Leibniz form.

Definition 4.1. Let C be a $\mathcal{VB}_{m,n}$ -gauge-natural operator in the sense of Definition 3.1. We say that C is a generalized twisted Dorfman–Courant bracket if it satisfies the initial condition $C_0 = [[-, -]]_0$, where $[[-, -]]_H$ is the usual twisted (H -twisted) Dorfman–Courant bracket as in Example 3.3.

As an immediate consequence of Theorem 3.6, we get

Lemma 4.2. Let $m, n \geq 1$ and $p \geq 1$ be natural numbers such that $m \geq p + 2$. Any generalized twisted Dorfman–Courant bracket C is of the form

$$(17) \quad \begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = & [X^1, X^2] \oplus \{ \mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 + \\ & + c_1 i_{X^1} i_{X^2} H + c_2 i_{L i_{X^1} i_{X^2}} dH \\ & + c_3 i_{L i_{X^2}} di_{X^1} H + c_4 i_{L i_{X^1}} di_{X^2} H \\ & + c_5 i_{L} di_{X^2} i_{X^1} H \} \end{aligned}$$

for any $H \in \Gamma_E^l(\bigwedge^{p+2} T^*E)$, any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus \bigwedge^p T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where c_1, c_2, c_3, c_4, c_5 are (uniquely determined by C) real numbers.

Definition 4.3. A $\mathcal{VB}_{m,n}$ -gauge-natural operator C in the sense of Definition 3.1 satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms if

$$(18) \quad C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3))$$

for all closed linear $(p+2)$ -forms $H \in \Gamma_E^l(\bigwedge^{p+2} T^*E)$, all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus \bigwedge^p T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$ -objects E .

Lemma 4.4. Let C be a generalized twisted Dorfman–Courant bracket of the form (17). If C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms, then

$$(19) \quad \begin{aligned} & c_3 \mathcal{L}_{X^1} i_{L i_{X^3}} di_{X^2} H + c_4 \mathcal{L}_{X^1} i_{L i_{X^2}} di_{X^3} H \\ & + c_5 \mathcal{L}_{X^1} i_{L} di_{X^3} i_{X^2} H + c_3 i_{L} i_{[X^2, X^3]} di_{X^1} H \\ & + c_4 i_{L} i_{X^1} di_{[X^2, X^3]} H + c_5 i_{L} di_{[X^2, X^3]} i_{X^1} H \\ & = -c_3 i_{X^3} di_{L i_{X^2}} di_{X^1} H - c_4 i_{X^3} di_{L i_{X^1}} di_{X^2} H \\ & - c_5 i_{X^3} di_{L} di_{X^2} i_{X^1} H + c_3 i_{L} i_{X^3} di_{[X^1, X^2]} H \\ & + c_4 i_{L} i_{[X^1, X^2]} di_{X^3} H + c_5 i_{L} di_{X^3} di_{[X^1, X^2]} H \\ & + c_3 \mathcal{L}_{X^2} i_{L i_{X^3}} di_{X^1} H + c_4 \mathcal{L}_{X^2} i_{L i_{X^1}} di_{X^3} H \\ & + c_5 \mathcal{L}_{X^2} i_{L} di_{X^3} i_{X^1} H + c_3 i_{L} i_{[X^1, X^3]} di_{X^2} H \\ & + c_4 i_{L} i_{X^2} di_{[X^1, X^3]} H + c_5 i_{L} di_{[X^1, X^3]} i_{X^2} H \end{aligned}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear $(p+2)$ -form H on $\mathbf{R}^{m,n}$.

Proof. For any linear vector fields X^1, X^2, X^3 and any closed linear $(p+2)$ -form H on E , we can write

$$\begin{aligned} C_H(X^1 \oplus 0, C_H(X^2 \oplus 0, X^3 \oplus 0)) &= [X^1, [X^2, X^3]] \oplus \Omega, \\ C_H(C_H(X^1 \oplus 0, X^2 \oplus 0), X^3 \oplus 0) &= [[X^1, X^2], X^3] \oplus \Theta, \\ C_H(X^2 \oplus 0, C_H(X^1 \oplus 0, X^3 \oplus 0)) &= [X^2, [X^1, X^3]] \oplus \mathcal{T}, \end{aligned}$$

where

$$\begin{aligned} \Omega &= c_1 \mathcal{L}_{X^1} i_{X^2} i_{X^3} H + c_3 \mathcal{L}_{X^1} i_L i_{X^3} di_{X^2} H \\ &\quad + c_4 \mathcal{L}_{X^1} i_L i_{X^2} di_{X^3} H + c_5 \mathcal{L}_{X^1} i_L di_{X^3} i_{X^2} H \\ &\quad + c_1 i_{X^1} i_{[X^2, X^3]} H + c_3 i_L i_{[X^2, X^3]} di_{X^1} H \\ &\quad + c_4 i_L i_{X^1} di_{[X^2, X^3]} H + c_5 i_L di_{[X^2, X^3]} i_{X^1} H, \\ \Theta &= -c_1 i_{X^3} di_{X^1} i_{X^2} H - c_3 i_{X^3} di_L i_{X^2} di_{X^1} H \\ &\quad - c_4 i_{X^3} di_L i_{X^1} di_{X^2} H - c_5 i_{X^3} di_L di_{X^2} i_{X^1} H \\ &\quad + c_1 i_{[X^1, X^2]} i_{X^3} H + c_3 i_L i_{X^3} di_{[X^1, X^2]} H \\ &\quad + c_4 i_L i_{[X^1, X^2]} di_{X^3} H + c_5 i_L di_{X^3} di_{[X^1, X^2]} H, \\ \mathcal{T} &= c_1 \mathcal{L}_{X^2} i_{X^1} i_{X^3} H + c_3 \mathcal{L}_{X^2} i_L i_{X^3} di_{X^1} H \\ &\quad + c_4 \mathcal{L}_{X^2} i_L i_{X^1} di_{X^3} H + c_5 \mathcal{L}_{X^2} i_L di_{X^3} i_{X^1} H \\ &\quad + c_1 i_{X^2} i_{[X^1, X^3]} H + c_3 i_L i_{[X^1, X^3]} di_{X^2} H \\ &\quad + c_4 i_L i_{X^2} di_{[X^1, X^3]} H + c_5 i_L di_{[X^1, X^3]} i_{X^2} H. \end{aligned}$$

Since C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms,

$$\Omega = \Theta + \mathcal{T}.$$

Moreover, the (usual) twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms. Indeed, the (usual) twisted Dorfman–Courant bracket is the restriction of the twisted Courant bracket (which satisfies the Jacobi identity in Leibniz form for closed $(p+2)$ -forms, see [3]). So, we have $\Omega = \Theta + \mathcal{T}$ in the case $c_3 = c_4 = c_5 = 0$, too. So, we have (19). \square

Lemma 4.5. *Let C be a generalized twisted Dorfman–Courant bracket of the form (17). Let $m, n \geq 1$ and $p \geq 1$ be such that $m \geq p + 3$. If C satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms, then $c_3 = c_4 = c_5 = 0$.*

Proof. Let $\tilde{\omega}_o := dx^3 \wedge \dots \wedge dx^{p+1}$ if $p \geq 2$ (then $\tilde{\omega}_o$ is well defined because $m \geq p + 1 \geq 3$) and $\tilde{\omega}_o := 1$ if $p = 1$. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$, $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and the closed linear $(p+2)$ -form $H := x^1 dx^1 \wedge dx^2 \wedge dy^1 \wedge \tilde{\omega}_o$ into (19), we get

$$\begin{aligned}
& c_3 \cdot 0 + c_4 \cdot (y^1 dx^1 \wedge \tilde{\omega}_o) + c_5 \cdot (y^1 dx^1 \wedge \tilde{\omega}_o) \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& = -c_3 \cdot y^1 dx^1 \wedge \tilde{\omega}_o - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^1 \wedge \tilde{\omega}_o) \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 \\
& + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0.
\end{aligned}$$

Hence $c_3 = -c_4$.

Similarly, let $\tilde{\omega}_o$ be as above. Putting linear vector fields $X^1 = x^2 \frac{\partial}{\partial x^1}$, $X^2 = \frac{\partial}{\partial x^2}$, $X^3 = L$ and the closed linear $(p+2)$ -form $H := dx^1 \wedge dx^2 \wedge dy^1 \wedge \tilde{\omega}_o$ into (19), we get

$$\begin{aligned}
& c_3 \cdot 0 + c_4 \cdot y^1 dx^2 \wedge \tilde{\omega}_o + c_5 \cdot y^1 dx^2 \wedge \tilde{\omega}_o \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& = -c_3 \cdot 0 - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^2 \wedge \tilde{\omega}_o) \\
& + c_3 \cdot 0 + c_4 \cdot y^1 dx^2 \wedge \tilde{\omega}_o + c_5 \cdot 0 \\
& + c_3 \cdot 0 + c_4 \cdot (-y^1 dx^2 \wedge \tilde{\omega}_o) + c_5 \cdot (-y^1 dx^2 \wedge \tilde{\omega}_o) \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0.
\end{aligned}$$

Hence $c_4 = -c_5$.

Now, let $\tilde{\omega}_o := dx^5 \wedge \dots \wedge dx^{p+3}$ if $p \geq 2$ (then $\tilde{\omega}_o$ is well defined because $m \geq p+3 \geq 5$), and $\tilde{\omega}_o := 1$ if $p = 1$. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$, $X^2 = x^1 \frac{\partial}{\partial x^2}$, $X^3 = \frac{\partial}{\partial x^3}$ and the closed linear $(p+2)$ -form $H := d(x^2 x^4) \wedge dx^3 \wedge dy^1 \wedge \tilde{\omega}_o$ (H is well defined because $m \geq p+3 \geq 4$) into (19), we get

$$\begin{aligned}
& c_3 \cdot y^1 dx^4 \wedge \tilde{\omega}_o + c_4 \cdot 0 + c_5 \cdot (-y^1 dx^4 \wedge \tilde{\omega}_o) \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& = -c_3 \cdot 0 - c_4 \cdot (y^1 dx^4 \wedge \tilde{\omega}_o + x^4 dy^1 \wedge \tilde{\omega}_o) - c_5 \cdot 0 \\
& + c_3 \cdot y^1 dx^4 \wedge \tilde{\omega}_o + c_4 \cdot 0 + c_5 \cdot 0 \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0.
\end{aligned}$$

Hence $c_4 = 0$.

Consequently, $c_3 = c_4 = c_5 = 0$, as well. \square

Thus we have proved

Theorem 4.6. *Let $m, n \geq 1$ and $p \geq 1$ be such that $m \geq p+3$. Any generalized twisted Dorfman–Courant bracket C satisfying the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms is of the form*

$$\begin{aligned}
(20) \quad C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= [X^1, X^2] \oplus \{\mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 \\
&+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH\}
\end{aligned}$$

for any $H \in \Gamma_E^l(\bigwedge^{p+2} T^*E)$, any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus \bigwedge^p T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where c_1, c_2 are (uniquely determined by C) real numbers.

Given $c_1, c_2 \in \mathbf{R}$, the generalized twisted Dorfman–Courant bracket C of the form (20) satisfies the Jacobi identity in Leibniz form for closed linear $(p+2)$ -forms.

The above theorem implies immediately Theorem 1.2.

REFERENCES

- [1] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.
- [2] Kurek, J., Mikulski, W. M., *The gauge-natural bilinear brackets on couples of linear vector fields and linear p -forms*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **75**(2) (2021), 73–92.
- [3] Mikulski, W. M., *The natural operators similar to the twisted Courant bracket on couples of vector fields and p -forms*, Filomat **44**(12) (2020), 4071–4078.
- [4] Mikulski, W. M., *On the gauge-natural operators similar to the twisted Dorfman–Courant bracket*, Opuscula Math **41**(2) (2021), 205–226.

Jan Kurek
 Institute of Mathematics
 Maria Curie-Skłodowska University
 Pl. Marii Curie-Skłodowskiej 1
 20-031 Lublin
 Poland
 e-mail: kurek@hektor.umcs.lublin.pl

Włodzimierz M. Mikulski
 Institute of Mathematics
 Jagiellonian University
 ul. Łojasiewicza 6
 30-348 Cracow
 Poland
 e-mail: Wlodzimierz.Mikulski@im.uj.edu.pl

Received February 7, 2022