

MILUTIN OBRADOVIĆ and NIKOLA TUNESKI

Some properties of the class \mathcal{U}

ABSTRACT. In this paper we study the class \mathcal{U} of functions that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized such that $f(0) = f'(0) - 1 = 0$ and satisfy

$$\left| \left[\frac{z}{f(z)} \right]^2 f'(z) - 1 \right| < 1 \quad (z \in \mathbb{D}).$$

For functions in the class \mathcal{U} we give sharp estimates of the second and the third Hankel determinant, its relationship with the class of α -convex functions, as well as certain starlike properties.

1. Introduction. Let \mathcal{A} denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let \mathcal{S}^* and \mathcal{K} denote the subclasses of \mathcal{A} which are starlike and convex in \mathbb{D} , respectively, i.e.,

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > 0, z \in \mathbb{D} \right\}$$

and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, z \in \mathbb{D} \right\}.$$

Geometrical characterisation of convexity is the usual one, while for the starlikeness we have $f \in \mathcal{S}^*$, if and only if $f(\mathbb{D})$ is a starlike region, i.e.,

$$z \in f(\mathbb{D}) \quad \Rightarrow \quad tz \in f(\mathbb{D}) \quad \text{for all } t \in [0, 1].$$

The linear combination of the expressions involved in the analytical representations of starlikeness and convexity brings us to the classes of α -convex functions introduced in 1969 by Mocanu [3] and consisting of functions $f \in \mathcal{A}$ such that

$$(1) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[1 + \frac{zf''(z)}{f'(z)} \right] \right\} > 0 \quad (z \in \mathbb{D}),$$

where $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Those classes he denoted by \mathcal{M}_α .

Further, let \mathcal{U} denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$|U_f(z)| < 1 \quad (z \in \mathbb{D}),$$

where the operator U_f is defined by

$$U_f(z) := \left[\frac{z}{f(z)} \right]^2 f'(z) - 1.$$

All these classes consist of univalent functions and more details on them can be found in [1, 10].

The class of starlike functions is very large and in the theory of univalent functions it is significant if a class does not entirely lie inside \mathcal{S}^* . One such case is the class of functions with bounded turning consisting of functions f from \mathcal{A} that satisfy $\operatorname{Re} f'(z) > 0$ for all $z \in \mathbb{D}$. Another example is the class \mathcal{U} defined above and first treated in [5] (see also [6, 7, 10]). Namely, the function $-\ln(1-z)$ is convex, thus starlike, but not in \mathcal{U} because $U_f(0.99) = 3.621 \dots > 1$, while the function f defined by

$$\frac{z}{f(z)} = 1 - \frac{3}{2}z + \frac{1}{2}z^3 = (1-z)^2 \left(1 + \frac{z}{2} \right)$$

is in \mathcal{U} and such that

$$\frac{zf'(z)}{f(z)} = -\frac{2(z^2 + z + 1)}{z^2 + z - 2} = -\frac{1}{5} + \frac{3i}{5}$$

for $z = i$. This property is the main reason why the class \mathcal{U} attracts huge attention in the past decades.

In this paper we give sharp estimates of the second and the third Hankel determinant over the class \mathcal{U} and study its relation with the class of α -convex and starlike functions.

2. Main results. In the first theorem we give the sharp estimates of the Hankel determinants of the second and third order for the class \mathcal{U} . We first give the definition of the Hankel determinant, whose elements are the coefficients of a function $f \in \mathcal{A}$.

Definition 2. Let $f \in \mathcal{A}$. Then the q th Hankel determinant of f is defined for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Thus, the second and the third Hankel determinants are, respectively,

$$(3) \quad \begin{aligned} H_2(2) &= a_2 a_4 - a_3^2, \\ H_3(1) &= a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \end{aligned}$$

Theorem 1. Let $f \in \mathcal{U}$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. Then we have the sharp estimates:

$$|H_2(2)| \leq 1 \quad \text{and} \quad |H_3(1)| \leq \frac{1}{4}.$$

Proof. In [5] the following characterization of functions f in the class in \mathcal{U} was given:

$$(4) \quad \frac{z}{f(z)} = 1 - a_2 z - z \int_0^z \frac{\omega(t)}{t^2} dt,$$

where function ω is analytic in \mathbb{D} with $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

If we put $\omega_1(z) = \int_0^z \frac{\omega(t)}{t^2} dt$, then we easily obtain $|\omega_1(z)| \leq |z| < 1$ and $|\omega_1'(z)| \leq 1$ for all $z \in \mathbb{D}$. If $\omega_1(z) = c_1 z + c_2 z^2 + \dots$, then $\omega_1'(z) = c_1 + 2c_2 z + 3c_3 z^2 + \dots$ and $|\omega_1'(z)| \leq 1$, $z \in \mathbb{D}$, gives (see relation (13) in the paper of Prokhorov and Szynal [8]):

$$(5) \quad |c_1| \leq 1, \quad |2c_2| \leq 1 - |c_1|^2 \quad \text{and} \quad |3c_3(1 - |c_1|^2) + 4\overline{c_1}c_2^2| \leq (1 - |c_1|^2)^2 - 4|c_2|^2.$$

Also, from (4) we have

$$\begin{aligned} f(z) &= \frac{z}{1 - (a_2 z + c_1 z^2 + c_2 z^3 + \dots)} \\ &= z + a_2 z^2 + (c_1 + a_2^2) z^3 + (c_2 + 2a_2 c_1 + a_2^3) z^4 \\ &\quad + (c_3 + 2a_2 c_2 + c_1^2 + 3a_2^2 c_1 + a_2^4) z^5 \dots \end{aligned}$$

From the last relation we have

$$(6) \quad a_3 = c_1 + a_2^2, \quad a_4 = c_2 + 2a_2 c_1 + a_2^3, \quad a_5 = c_3 + 2a_2 c_2 + c_1^2 + 3a_2^2 c_1 + a_2^4.$$

We may suppose that $c_1 \geq 0$, since from (6) we have $c_1 = a_3 - a_2^2$ and a_3 and a_2^2 have the same turn under rotation. In that sense, from (5) we obtain

$$(7) \quad 0 \leq c_1 \leq 1, \quad |c_2| \leq \frac{1}{2}(1 - c_1^2) \quad \text{and} \quad |c_3| \leq \frac{1}{3} \left(1 - c_1^2 - \frac{4|c_2|^2}{1 + c_1} \right).$$

If we use (3), (6) and (7), then

$$\begin{aligned} |H_2(2)| &= |c_2 a_2 - c_1^2| \leq |c_2| \cdot |a_2| + c_1^2 \leq \frac{1}{2} (1 - c_1^2) |a_2| + c_1^2 \\ &= \frac{1}{2} \cdot |a_2| + \left(1 - \frac{1}{2} \cdot |a_2|\right) c_1^2 \leq 1. \end{aligned}$$

The functions $k(z) = \frac{z}{(1-z)^2}$ and $f_1(z) = \frac{z}{1-z^2}$ show that the estimate is the best possible.

Similarly, after some calculations we also have

$$\begin{aligned} |H_3(1)| &= |c_1 c_3 - c_2^2| \leq c_1 |c_3| + |c_2|^2 \\ &\leq \frac{1}{3} c_1 \left(1 - c_1^2 - \frac{4|c_2|^2}{1+c_1}\right) + |c_2|^2 \\ &= \frac{1}{3} \left(c_1 - c_1^3 + \frac{3-c_1}{1+c_1} |c_2|^2\right) \\ &= \frac{1}{3} \left(c_1 - c_1^3 + \frac{3-c_1}{1+c_1} \cdot \frac{1}{4} (1-c_1^2)^2\right) \\ &= \frac{1}{12} (3 - 2c_1^2 - c_1^4) \leq \frac{3}{12} = \frac{1}{4}. \end{aligned}$$

The function $f_2(z) = \frac{z}{1-z^{3/2}}$ shows that the result is the best possible. \square

In the rest of the paper we consider some starlikeness problems for the class \mathcal{U} and its connection with the class of α -convex functions.

First, let us recall the classical results about the relation between the starlike functions and α -convex functions.

Theorem 2.

- (a) $\mathcal{M}_\alpha \subseteq \mathcal{S}^*$ for every real α ([4]);
- (b) for $0 \leq \frac{\beta}{\alpha} \leq 1$ we have $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$ and for $\alpha > 1$, $\mathcal{M}_\alpha \subset \mathcal{M}_1 = \mathcal{K}$ ([9, 4]).

As an analogue of the above theorem we have

Theorem 3. For the classes \mathcal{M}_α the next results are valid.

- (a) $\mathcal{M}_\alpha \subset \mathcal{U}$ for $\alpha \leq -1$;
- (b) \mathcal{M}_α is not a subset of \mathcal{U} for any $0 \leq \alpha \leq 1$.

Proof. (a) Let $p(z) = U_f(z)$. Then p is analytic in \mathbb{D} and $p(0) = p'(0) = 0$.

From this we have $\left[\frac{z}{f(z)}\right]^2 f'(z) = p(z) + 1$ and, after some calculations,

$$2 \frac{z f'(z)}{f(z)} - \left[1 + \frac{z f''(z)}{f'(z)}\right] = 1 - \frac{z p'(z)}{p(z) + 1}.$$

The relation (1) is equivalent to

$$(8) \quad \operatorname{Re} \left\{ (1 + \alpha) \frac{zf'(z)}{f(z)} - \alpha \left[1 - \frac{zp'(z)}{p(z) + 1} \right] \right\} > 0, \quad z \in \mathbb{D}.$$

We want to prove that $|p(z)| < 1$, $z \in \mathbb{D}$. If not, then according to the Clunie–Jack Lemma ([2]) there exists a z_0 , $|z_0| < 1$, such that $p(z_0) = e^{i\theta}$ and $z_0 p'(z_0) = kp(z_0) = ke^{i\theta}$, $k \geq 2$. For such z_0 , from (8) we get

$$\begin{aligned} & \operatorname{Re} \left\{ (1 + \alpha) \frac{z_0 f'(z_0)}{f(z_0)} - \alpha \left[1 - \frac{ke^{i\theta}}{e^{i\theta} + 1} \right] \right\} \\ &= (1 + \alpha) \operatorname{Re} \left[\frac{z_0 f'(z_0)}{f(z_0)} \right] + \alpha \frac{k - 2}{2} \leq 0 \end{aligned}$$

since $f \in \mathcal{S}^*$ (by Theorem 2) and $\alpha \leq -1$. That is a contradiction to (1). It means that $|p(z)| = |U_f(z)| < 1$, $z \in \mathbb{D}$, i.e., $f \in \mathcal{U}$.

(b) To prove this part, by using Theorem 2(b), it is enough to find a function $g \in \mathcal{K}$ such that g does not belong to the class \mathcal{U} . Really, the function $g(z) = -\ln(1 - z)$ is convex but not in \mathcal{U} . \square

Open problem. It remains an open problem to study the relationship between classes \mathcal{M}_α and \mathcal{U} when $-1 < \alpha < 0$ and $\alpha > 1$.

In the next theorem we consider starlikeness of the function

$$(9) \quad g(z) = \frac{z/f(z) - 1}{-a_2},$$

where $f \in \mathcal{U}$ and $a_2 = \frac{f''(0)}{2} \neq 0$, i.e., its second coefficient does not vanish.

Namely, we have

Theorem 4. *Let $f \in \mathcal{U}$. Then, for the function g defined by (9) we have:*

- (a) $|g'(z) - 1| < 1$ for $|z| < |a_2|/2$;
- (b) $g \in \mathcal{S}^*$ in the disk $|z| < |a_2|/2$ and even more

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| < 1 \quad (|z| < |a_2|/2);$$

- (c) $g \in \mathcal{U}$ in the disk $|z| < |a_2|/2$ if $0 < |a_2| \leq 1$.

The results are best possible.

Proof. Let $f \in \mathcal{U}$ with $a_2 \neq 0$. Then, by using (4), we have

$$\frac{z}{f(z)} = 1 - a_2 z - z\omega_1(z),$$

where ω_1 is analytic in \mathbb{D} such that $|\omega_1(z)| \leq |z|$ and $|\omega_1'(z)| \leq 1$. The appropriate function g from (9) has the form

$$g(z) = z + \frac{1}{a_2} z\omega_1(z).$$

From here $|g'(z) - 1| = \frac{1}{|a_2|} |\omega_1(z) + z\omega_1'(z)| < 1$ for $|z| < |a_2|/2$.

By using previous representation, we obtain

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| = \left| \frac{z\omega_1'(z)}{a_2 + \omega_1(z)} \right| \leq \frac{|z|}{|a_2| - |z|} < 1$$

if $|z| < |a_2|/2$. It means that the function g is starlike in the disk $|z| < |a_2|/2$.

If we consider function f_b defined by

$$(10) \quad \frac{z}{f_b(z)} = 1 + bz + z^2, \quad 0 < b \leq 2,$$

then $f_b \in \mathcal{U}$ and

$$g_b(z) = \frac{\frac{z}{f_b(z)} - 1}{b} = z + \frac{1}{b}z^2.$$

For this function we can easily see that for $|z| < b/2$,

$$\operatorname{Re} \frac{zg_b'(z)}{g_b(z)} \geq \frac{1 - \frac{2}{b}|z|}{1 - \frac{1}{b}|z|} > 0.$$

On the other hand, since $g_b'(-b/2) = 0$, the function g_b is not univalent in a bigger disk, which implies that our result is best possible.

Also, by using (9) and the next estimation for the function ω_1 :

$$|z\omega_1'(z) - \omega_1(z)| \leq \frac{r^2 - |\omega_1(z)|^2}{1 - r^2},$$

(where $|z| = r$ and $|\omega_1(z)| \leq r$), after some calculation, we get

$$\begin{aligned} |\mathcal{U}_g(z)| &= \left| \frac{\frac{1}{a_2}(z\omega_1'(z) - \omega_1(z)) - \frac{1}{a_2}\omega_1^2(z)}{\left(1 + \frac{1}{a_2}\omega_1(z)\right)^2} \right| \\ &\leq \frac{|a_2||z\omega_1'(z) - \omega_1(z)| + |\omega_1(z)|^2}{(|a_2| - |\omega_1(z)|)^2} \\ &\leq \frac{|a_2|\frac{r^2 - |\omega_1(z)|^2}{1 - r^2} + |\omega_1(z)|^2}{(|a_2| - |\omega_1(z)|)^2} \\ &=: \frac{1}{1 - r^2}\varphi(t), \end{aligned}$$

where we put

$$\varphi(t) = \frac{(1 - r^2 - |a_2|)t^2 + |a_2|r^2}{(|a_2| - t)^2}$$

and $|\omega_1(z)| = t$, $0 \leq t \leq r$. We have

$$\begin{aligned} \varphi'(t) &= \frac{2|a_2|}{(|a_2| - t)^3} ((1 - r^2 - |a_2|)t + r^2) \\ &= \frac{2|a_2|}{(|a_2| - t)^3} ((1 - |a_2|)t + (1 - t)r^2) \geq 0, \end{aligned}$$

because $0 < |a_2| \leq 1$ and $0 \leq t < 1$. It means that φ is an increasing function and

$$\varphi(t) \leq \varphi(r) = \frac{(1-r^2)r^2}{(|a_2|-r)^2}.$$

Finally, we have

$$|U_g(z)| \leq \frac{r^2}{(|a_2|-r)^2} < 1,$$

since $|z| < |a_2|/2$. This implies the second statement of the theorem.

As for sharpness, we can also consider the function f_b defined by (10) with $0 < b \leq 1$. For $|z| < \frac{b}{2}$ we have

$$|U_{g_b}(z)| \leq \frac{\frac{1}{b^2}|z|^2}{(1-\frac{1}{b}|z|)^2} < 1,$$

which implies that g_b belongs to the class \mathcal{U} in the disk $|z| < b/2$. \square

We believe that part (b) of the previous theorem is valid for all $0 < |a_2| \leq 2$. In that sense we have the next

Conjecture 1. *Let $f \in \mathcal{U}$. Then the function g defined by the expression (9) belongs to the class \mathcal{U} in the disk $|z| < |a_2|/2$. The result is the best possible.*

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Milutin Obradović
Department of Mathematics
Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73
11000 Belgrade
Serbia
e-mail: obrad@grf.bg.ac.rs

Nikola Tuneski
Department of Mathematics and Informatics
Faculty of Mechanical Engineering
Ss. Cyril and Methodius University in Skopje
Karpoš II b.b.
1000 Skopje
Republic of North Macedonia
e-mail: nikola.tuneski@mf.edu.mk

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