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A Note on Sibley Type Metrics

Uwaga o metrykach typu Sibley'a

Abstract. J. W. Thompson [3] has shown in what way the Levy's distance changes under the scale changing. We modify those inequalities such that they are satisfied for Sibley's and Sibley-Prokhorov's metrics.

1. **Introduction.** Let L be the Lévy distance and let F_α, $\alpha > 0$, denote the function defined by $F_\alpha(x) = F(x/\alpha)$, where F denotes a distribution function.

The following inequalities were given by Thompson [3]:

$$\text{(1)}\quad \begin{aligned} &\alpha L(F_1,G_1) \le L(F_\alpha,G_\alpha) \le L(F_1,G_1)\,, \quad \alpha < 1\,, \\ &L(F_1,G_1) \le L(F_\alpha,G_\alpha) \le \alpha L(F_1,G_1)\,, \quad \alpha > 1\,. \end{aligned}$$

A similar result holds for the Prokhorov distance Π :

$$\text{(2)}\quad \begin{aligned} &\alpha\Pi(P_1,Q_1) \le \Pi(P_\alpha,Q_\alpha) \le \Pi(P_1,Q_1)\,, \quad \alpha < 1 \\ &\Pi(P_1,Q_1) \le \Pi(P_\alpha,Q_\alpha) \le \alpha\Pi(P_1,Q_1)\,, \quad \alpha > 1\,, \end{aligned}$$

where $P_\alpha(A) = P(\alpha^{-1}A)$, $\alpha > 0$, and P is a probability measure.

It has been proved that these inequalities are not satisfied for the Sibley and the Sibley-Prokhorov distances (cf. [2]). This note shows that, under some appropriate modifications, (1) and (2) hold for the Sibley and Sibley-Prokhorov distances.

2. **Generalization of the Sibley and the Prokhorov distances.** Let $\mathcal{F}$ be the family of functions $F : \mathbb{R} \to [0,1]$ nondecreasing and left continuous. By the generalized Sibley function we mean the function $L^\theta_s : \mathcal{F} \times \mathcal{F} \to [0, (\sqrt{2}\sin\theta)^{-1})]$, $0 < \theta < \pi/2$, such that

$$\begin{aligned} L^\theta_s(F,G) = \inf\{ h > 0 : &F(x - \sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta \le G(x) \le \\ &\le F(x + \sqrt{2}h\cos\theta) + \sqrt{2}h\sin\theta\,, G(x - \sqrt{2}h\cos\theta) - \sqrt{2}h\sin\theta \le F(x) \le \\ &G(x + \sqrt{2}h\cos\theta) + \sqrt{2}h\sin\theta\,,\ |x| < 1/h\}\,. \end{aligned}$$

If $\theta = \pi/4$ we get the Sibley metric L_s.

Let $\mathcal{X}$ be a normed linear space with the norm $\| \ \|$ and $\mathcal{B}_{\mathcal{X}}$ – σ–field of subsets of $\mathcal{X}$. Denote by $\mathcal{P}$ the space of all probability measures on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ together with defective probability measures i.e. $P \in \mathcal{P}$ iff $P(\mathcal{X}) \leq 1$. Let $K(r)$ stand for the ball $K(r) = \{x \in \mathcal{X} : \|x\| < r\}$.

By the generalized Sibley–Prokhorov function we mean the function $\Pi_s^\theta : \mathcal{P} \times \mathcal{P} \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \pi/2$, such that

$$\Pi_s^\theta(P,Q) = \inf\{h > 0 : P(A) \leq Q(A^{\sqrt{2}h\cos\theta}) + \sqrt{2}h\sin\theta\,,$$
$$Q(A) \leq P(A^{\sqrt{2}h\cos\theta}) + \sqrt{2}h\sin\theta\,,\ A \in \mathcal{B}_{\mathcal{X}},\ A \subset K(1/h)\}\,,$$

where $A^r = \{x \in \mathcal{X} : \mathrm{dist}(x, A) < r\}$.

This function reduces to the Sibley–Prokhorov distance Π_s, when $\theta = \pi/4$.

It is not difficult to prove (cf. [2]) the following results :

Theorem 1.

$$(3)\quad (\sqrt{2}\cos\theta)^{-1}L_s(F,G) \leq L_s^\theta(F,G) \leq (\sqrt{2}\sin\theta)^{-1}L_s(F,G)\,, \quad 0 < \theta < \pi/4\,,$$
$$(4)\quad (\sqrt{2}\sin\theta)^{-1}L_s(F,G) \leq L_s^\theta(F,G) \leq (\sqrt{2}\cos\theta)^{-1}L_s(F,G)\,, \quad \pi/4 < \theta < \pi/2\,.$$

Theorem 2.

$$(5)\quad (\sqrt{2}\cos\theta)^{-1}\Pi_s(P,Q) \leq \Pi_s^\theta(P,Q) \leq (\sqrt{2}\sin\theta)^{-1}\Pi_s(P,Q)\,, \quad 0 < \theta < \pi/4\,,$$
$$(6)\quad (\sqrt{2}\sin\theta)^{-1}\Pi_s(P,Q) \leq \Pi_s^\theta(P,Q) \leq (\sqrt{2}\cos\theta)^{-1}\Pi_s(P,Q)\,, \quad \pi/4 < \theta < \pi/2\,.$$

To abbreviate the notation we shall write

$$\{h > 0 : F(x-h) - h \leq G(x) \leq F(x+h) + h,\ I(F \Longleftrightarrow G),\ |x| < 1/h\}$$

instead of

$$\{h > 0 : F(x-h) - h \leq G(x) \leq F(x+h) + h,\ G(x-h) - h \leq F(x) \leq G(x+h) + h\,,$$
$$|x| < 1/h\}\,,$$

i.e. $I(F \Longleftrightarrow G)$ denotes the preceding inequality with F replaced by G and vice versa.

3. Results.

Theorem 3. *Let $\alpha > 0$. If $\alpha < 1$, then*

$$(7)\qquad \alpha L_s(F_1, G_1) \leq L_s(F_\alpha, G_\alpha) \leq (\alpha\sqrt{2}\sin\arctan\alpha)^{-1}L_s(F_1, G_1)\,,$$

if $\alpha \geq 1$, then

$$(8)\qquad (\alpha\sqrt{2}\sin\arctan\alpha)^{-1}L_s(F_1, G_1) \leq L_s(F_\alpha, G_\alpha) \leq \alpha L_s(F_1, G_1)\,.$$

Proof. Let $\tan\theta = \alpha$. By the Definition of L_s we have

$$L_s(F_\alpha, G_\alpha) = \inf\{h > 0 : F_\alpha(x-h) - h \le G_\alpha(x) \le F_\alpha(x+h) + h,$$
$$I(F_\alpha \Longleftrightarrow G_\alpha),\ |x| < 1/h\} =$$
$$= \inf\{h > 0 : F_1(x/\alpha - h/\alpha) - h \le G_1(x/\alpha) \le F_1(x/\alpha + h/\alpha) + h,$$
$$I(F_1 \Longleftrightarrow G_1),\ |x| < 1/h\} =$$
$$= \inf\{h > 0 : F_1(y - h/\alpha) - h \le G_1(y) \le F_1(y + h/\alpha) + h,$$
$$I(F_1 \Longleftrightarrow G_1),\ |y| < 1/(\alpha h)\},$$

where $y = x/\alpha$. Putting $\alpha = \sin\theta/\cos\alpha$ and $h = \sqrt{2}h'\sin\theta$, we get

$$(9)\qquad L_s(F_\alpha, G_\alpha) = \inf\{\sqrt{2}h'\sin\theta > 0 : F_1(y - \sqrt{2}h'\cos\theta) - \sqrt{2}h'\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}h'\cos\theta) + \sqrt{2}h'\sin\theta,\ I(F_1 \Longleftrightarrow G_1),$$
$$|y| < 1/(\alpha\sqrt{2}h'\sin\theta)\} =$$
$$= \sqrt{2}\sin\theta \inf\{h' > 0 : F_1(y - \sqrt{2}h'\cos\theta) - \sqrt{2}h'\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}h'\cos\theta) + \sqrt{2}h'\sin\theta,\ I(F_1 \Longleftrightarrow G_1),$$
$$|y| < 1/(\alpha\sqrt{2}h'\sin\theta)\}\ .$$

If $0 < \alpha < 1$, then $0 < \theta < \pi/4$ and $1/h' < 1/(\alpha\sqrt{2}h'\sin\theta)$. Therefore,

$$(10)\quad (\sqrt{2}\sin\theta)^{-1} L_s(F_\alpha, G_\alpha) \ge \inf\{h' > 0 : F_1(y - \sqrt{2}h'\cos\theta) - \sqrt{2}h'\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}h'\cos\theta) + \sqrt{2}h'\sin\theta,\ I(F_1 \Longleftrightarrow G_1),\ |y| < 1/h'\} =$$
$$= L_s^\theta(F_1, G_1)\ .$$

Thus by (9) and (10) we have

$$\sqrt{2}\sin\theta L_s^\theta(F_1, G_1) \le L_s(F_\alpha, G_\alpha)\ .$$

Using the inequality (3) we obtain

$$\alpha L_s(F_1, G_1) \le \sqrt{2}\sin\theta L_s^\theta(F_1, G_1) \le L_s(F_\alpha, G_\alpha)\ ,$$

which proves the left hand side of the inequality (7).

Now, taking into account that $\alpha\sqrt{2}\sin\theta < 1$, and next putting $h' = h''/[\alpha\sqrt{2}\sin\theta]$, we have

$$(11)\ (\sqrt{2}\sin\theta)^{-1} L_s(F_\alpha, G_\alpha) = \inf\{h' > 0 : F_1(y - \sqrt{2}h'\cos\theta) - \sqrt{2}h'\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}h'\cos\theta) + \sqrt{2}h'\sin\theta,$$
$$I(F_1 \Longleftrightarrow G_1),\ |y| < 1/(\alpha\sqrt{2}h'\sin\theta)\} \le$$
$$\le \inf\{h' > 0 : F_1(y - \sqrt{2}\alpha\sqrt{2}\sin\theta h'\cos\theta) - \sqrt{2}\alpha\sqrt{2}\sin\theta h'\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}\alpha\sqrt{2}\sin\theta h'\cos\theta) + \sqrt{2}\alpha\sqrt{2}\sin\theta h'\sin\theta,$$
$$I(F_1 \Longleftrightarrow G_1),\ |y| < 1/(\alpha\sqrt{2}h'\sin\theta)\} =$$
$$= (\alpha\sqrt{2}\sin\theta)^{-1} \inf\{h'' > 0 : F_1(y - \sqrt{2}h''\cos\theta) - \sqrt{2}h''\sin\theta \le$$
$$\le G_1(y) \le F_1(y + \sqrt{2}h''\cos\theta) + \sqrt{2}h''\sin\theta,$$
$$I(F_1 \Longleftrightarrow G_1),\ |y| < 1/h''\} =$$
$$= (\alpha\sqrt{2}\sin\theta)^{-1} L_s^\theta(F_1, G_1)\ .$$

Using (9) and (11) we get

$$L_s(F_\alpha, G_\alpha) \leq \alpha^{-1} L_s^\theta(F_1, G_1) .$$

By (3) we obtain

$$L_s(F_\alpha, G_\alpha) \leq \alpha^{-1} L_s^\theta(F_1, G_1) \leq (\alpha\sqrt{2}\sin\theta)^{-1} L_s(F_1, G_1) .$$

But $\tan\theta = \alpha$, then

$$L_s(F_\alpha, G_\alpha) \leq (\alpha\sqrt{2}\sin\arctan\alpha)^{-1} L_s(F_1, G_1) ,$$

which proves the right hand side of the inequality (7).

The proof of the inequality (8) is similar after using (4).

Theorem 4. *Let $\alpha > 0$. If $\alpha < 1$, then*

$$\alpha\Pi_s(P_1, Q_1) \leq \Pi_s(P_\alpha, Q_\alpha) \leq (\alpha\sqrt{2}\sin\arctan\alpha)^{-1}\Pi_s(P_1, Q_1) , \tag{12}$$

if $\alpha \geq 1$, then

$$(\alpha\sqrt{2}\sin\arctan\alpha)^{-1}\Pi_s(P_1, Q_1) \leq \Pi_s(P_\alpha, Q_\alpha) \leq \alpha\Pi_s(P_1, Q_1) . \tag{13}$$

Proof. Let $\tan\theta = \alpha$. By definition of Π_s, we have

$$\begin{aligned}\Pi_s(P_\alpha, Q_\alpha) &= \inf\{h > 0 : P_\alpha(A) \leq Q_\alpha(A^h) + h,\ I(P_\alpha \Longleftrightarrow Q_\alpha),\ A \subset K(1/h)\} = \\ &= \inf\{h > 0 : P_1(\alpha^{-1}A) \leq Q_1(\alpha^{-1}A^h) + h,\ I(P_1 \Longleftrightarrow Q_1),\ A \subset K(1/h)\} = \\ &= \inf\{h > 0 : P_1(\alpha^{-1}A) \leq Q_1((\alpha^{-1}A)^{h/\alpha}) + h,\ I(P_1 \Longleftrightarrow Q_1),\ A \subset K(1/h)\} =\end{aligned}$$

as $\{x : x \in \alpha^{-1}A^h\} = \{x : x \in (\alpha^{-1}A)^{h/\alpha}\}$.

Putting $\alpha^{-1}A = B$, $\alpha = \sin\theta/\cos\theta$, and $h = \sqrt{2}h'\sin\theta$, we obtain

$$\begin{aligned}\Pi_s(P_\alpha, Q_\alpha) &= \inf\{h > 0 : P_1(B) \leq Q_1(B^{h/\alpha}) + h,\ I(P_1 \Longleftrightarrow Q_1),\ B \subset K(1/\alpha h)\} = \\ &= \inf\{\sqrt{2}h'\sin\theta > 0 : P_1(B) \leq Q_1(B^{\sqrt{2}h'\cos\theta}) + \sqrt{2}h'\sin\theta, \\ &\qquad I(P_1 \Longleftrightarrow Q_1),\ B \subset K(1/[\alpha\sqrt{2}h'\sin\theta])\} = \\ &= \sqrt{2}\sin\theta \inf\{h' > 0 : P_1(B) \leq Q_1(B^{\sqrt{2}h'\cos\theta}) + \sqrt{2}h'\sin\theta, \\ &\qquad I(P_1 \Longleftrightarrow Q_1),\ B \subset K(1/[\alpha\sqrt{2}h'\sin\theta])\} .\end{aligned}$$

If $0 < \alpha < 1$, then $0 < \theta < \pi/4$ and $1/h' < 1/(\alpha\sqrt{2}h'\sin\theta)$. Therefore,

$$\begin{aligned}(\sqrt{2}\sin\theta)^{-1}\Pi_s(P_\alpha, Q_\alpha) &\geq \inf\{h' > 0 : P_1(B) \leq Q_1(B^{\sqrt{2}h'\cos\theta}) + \sqrt{2}h'\sin\theta, \\ &\quad I(P_1 \Longleftrightarrow Q_1),\ B \subset K(1/h')\} = \Pi_s^\theta(P_1, Q_1) .\end{aligned}$$

Thus, we have

$$\sqrt{2}\sin\theta\Pi_s^\theta(P_1,Q_1) \le \Pi_s(P_\alpha,Q_\alpha) .$$

Using (5), we get

$$\alpha\Pi_s(P_1,Q_1) \le \sqrt{2}\sin\theta\Pi_s^\theta(P_1,Q_1) \le \Pi_s(P_\alpha,Q_\alpha) ,$$

which proves the left hand side of the inequality (12).

Similar considerations lead us to the right hand side of inequality (12). The estimates (13) can be done in the same way after using (6).

REFERENCES

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STRESZCZENIE

J.W. Thompson [3] pokazał jak zmienia się odległość Lévy'ego między dystrybuantami przy zmianie skali. W niniejszej pracy zmodyfikowano te nierówności w ten sposób, aby były one słuszne dla metryk Sibley'a i Sibley'a–Prochorowa.