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On a Decomposition of the Riemannian Manifold with a 3-structure

O rozkładzie rozmaitości Riemanna z 3-strukturą

Abstract. This paper deals with a representation of a $4n$-dimensional Riemannian manifold as a Cartesian product of $\mathbf{R}^n$ and three n-dimensional manifolds with suitably chosen structures. These structures generate a 3-structure on the $4n$-dimensional manifold and the corresponding construction is based of the notion of vector field and distribution.

Also integrability conditions of intervening distributions are given.

Introduction. This paper is a continuation of the study on the Riemannian manifolds M^{4n} with the 3-structure satisfying certain conditions, cf. [1]–[3]. On the basis of the results shown in the quoted papers, the Riemannian manifold was obtained in the form of the Cartesian product $M_1^n \times M_2^n \times M_3^n \times R^n$, where M_1^n, M_2^n, M_3^n are the manifolds of the corresponding structures $\underset{1}{G}, \underset{2}{G}, \underset{3}{G}$ satisfying corresponding relations. Each of the manifolds is invariant in relation to a corresponding structure. As the whole construction is based first of all on the notion of vector fields and distribution. Conditions of integrability of corresponding distributions were studied to obtain the required decomposition.

1. Properties of the tangent bundle to Riemannian manifold with a 3-structure. Let M^{4n} be the Riemannian manifold of the dimension $4n$ with the metric $\bar g$ and with the generalized 3-structure $\{\underset{\alpha}{\bar F}\}$ satisfying the conditions :

$$(1)\qquad \begin{aligned} (\underset{\alpha}{\bar F})^2 &= \underset{\alpha}{\varepsilon}\bar I \\ \underset{\alpha}{\bar F}\circ\underset{\beta}{\bar F} &= \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\bar F} \end{aligned}$$

$\alpha,\beta,\gamma = 1,2,3$, $\alpha \neq \beta \neq \gamma \neq \alpha$, $\underset{\alpha}{\varepsilon} = \pm 1$, $\underset{\alpha\beta}{\varepsilon} = \pm 1$, $\underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon} = \underset{\alpha\beta}{\varepsilon}\underset{\gamma}{\varepsilon}$, $\underset{\alpha\beta}{\varepsilon}\underset{\alpha\gamma}{\varepsilon} = \underset{\beta\alpha}{\varepsilon}\underset{\gamma\alpha}{\varepsilon} = \underset{\alpha}{\varepsilon}$, $\underset{\alpha}{\bar F}$ are tensor fields of the type (1.1) on M^{4n}, $\bar I$ – the identity mapping on M^{4n} ([1]). Moreover, it is assumed that

$$(2)\qquad \bar g(\underset{\alpha}{\bar F}\bar X, \underset{\alpha}{\bar F}\bar Y) = \bar g(\bar X, \bar Y)$$

for any vector fields, $\tilde{X}, \tilde{Y} \in TM^{4n}$, TM^{4n} – the tangent bundle on M^{4n}.

Let us introduce the notation: $TM^{4n} = \tilde{V}_1$. Let N_1 be a given differentiable vector field on M^{4n} , $N_1 \in TM^{4n}$ and $\tilde{g}(N_1, N_1) = 1$. We denote by V_1 , $V_1 \subset \tilde{V}_1$ the distribution on M^{4n} orthogonal to the field N_1 with respect to the metric $\tilde{g}$. Thus : $\dim \tilde{V}_1 = 4n$, $\dim V_1 = 4n - 1$.

For any field $\tilde{X} \in \tilde{V}_1$ we put

$$\underset{\alpha}{\tilde{F}}\tilde{X} = \underset{\alpha}{X_1} + \underset{\alpha}{X_2}\,, \tag{3}$$

where $\underset{\alpha}{X_1} \in V_1$, $\underset{\alpha}{X_2} \in \{N_1\}$. Let us denote

$$\underset{\alpha}{X_1} = \underset{\alpha}{F^1}\tilde{X}\,, \qquad \underset{\alpha}{X_2} = \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega^1}(\tilde{X})N_1\,,$$

where $\underset{\alpha}{F^1} : \tilde{V}_1 \to V_1$ – the tensor field of the type (1.1) on M^{4n} , $\underset{\alpha}{\omega^1} : \tilde{V}_1 \to R$ – the tensor field of the type (0,1). We have so the decomposition

$$\underset{\alpha}{\tilde{F}}\tilde{X} = \underset{\alpha}{F^1}\tilde{X} + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega^1}(\tilde{X})N_1. \tag{3'}$$

Particularly, for the vector field $X \in V_1$ there is

$$\underset{\alpha}{\tilde{F}}X = \underset{\alpha}{F^1}X + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega^1}(X)N_1\,, \tag{4}$$

but for the vector field N_1 :

$$\underset{\alpha}{\tilde{F}}N_1 = \underset{\alpha}{\eta^1} + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda^1}N_1\,, \tag{5}$$

where $\underset{\alpha}{\eta^1} = \underset{\alpha}{F^1}N_1 \in V_1$, $\underset{\alpha}{\lambda^1} = \underset{\alpha}{\omega^1}(N_1) \in R$. It follows from the Theorem 2,[1] that for the distribution V_1 the following conditions are satisfied :

$$\begin{cases} (\underset{\alpha}{F^1})^2 = \underset{\alpha}{\varepsilon}(I - \underset{\alpha}{\omega^1} \otimes \underset{\alpha}{\eta^1}) \\ \underset{\alpha}{\omega^1} \circ \underset{\alpha}{F^1} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda^1}\,\underset{\alpha}{\omega^1} \\ \underset{\alpha}{F^1}\,\underset{\alpha}{\eta^1} = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda^1}\,\underset{\alpha}{\eta^1} \\ \underset{\alpha}{\omega^1}(\underset{\alpha}{\eta^1}) = 1 - \underset{\alpha}{\varepsilon}(\underset{\alpha}{\lambda^1})^2 \end{cases} \tag{6}$$

and

$$\begin{cases} \underset{\alpha}{F^1} \circ \underset{\beta}{F^1} = \underset{\alpha\beta}{\varepsilon}{}^{\gamma}\,\underset{\gamma}{F^1} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\omega^1} \otimes \underset{\alpha}{\eta^1} \\ \underset{\alpha}{\omega^1} \circ \underset{\beta}{F^1} = \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\omega^1} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda^1}\,\underset{\beta}{\omega^1} \\ \underset{\alpha}{F^1}\,\underset{\beta}{\eta^1} = \underset{\alpha\beta}{\varepsilon}{}^{\gamma}\,\underset{\gamma}{\eta^1} - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda^1}\,\underset{\alpha}{\eta^1} \\ \underset{\alpha}{\omega^1}(\underset{\beta}{\eta^1}) = \underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda^1} - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda^1}\,\underset{\beta}{\lambda^1} \end{cases} \tag{7}$$

Moreover, for the vector fields $X, Y \in V_1$ the following relations take place :

$$(8) \qquad \begin{cases} g^1(X, \underset{\alpha}{\eta}^1) = \underset{\alpha}{\omega}^1(X) \\ g^1(\underset{\alpha}{F}^1 X, \underset{\alpha}{F}^1 Y) = g^1(X,Y) - \underset{\alpha}{\omega}^1(X)\underset{\alpha}{\omega}^1(Y) , \end{cases}$$

where g^1 is the metric induced on the distribution V_1 i.e.

$$g^1(X,Y) = \tilde{g}(X,Y) , \qquad X, Y \in V_1 ,$$

(Theorem 4,[1]) .

Let us assume that the fields $\underset{1}{\eta}^1, \underset{2}{\eta}^1, \underset{3}{\eta}^1 \in V_1$ are linearly independent. Conditions for existence of such fields were given in the paper [1], Theorem 3. Let $W_1 = \mathrm{Lin}(\underset{1}{\eta}^1, \underset{2}{\eta}^1, \underset{3}{\eta}^1)$. Let $W_1^{\perp} \subset V_1$ denote orthogonal complement to the distribution $W_1 \subset V_1$ in relation to the metric g^1. Thus, $\dim W_1 = 3$, $\dim W_1 = 4n - 4 = 4(n-1)$ and

$$\tilde{V}_1 = \{N_1\} \oplus W_1 \oplus W_1^{\perp} , \qquad \{N_1\} = \mathrm{Lin}\, N_1$$
$$V_1 = W_1 \oplus W_1^{\perp}$$

where the above distribution means that $(\tilde{V}_1)_z = (\{N_1\})_z + (W_1)_z + (W_1^{\perp})_z$ and $(V_1)_z = (W_1)_z + (W_1^{\perp})_z$ for each $z \in M^{4n}$.

For any field $X \in W_1^{\perp}$ from (8) there takes place

$$(9) \qquad \underset{\alpha}{\omega}^1(X) = 0 , \qquad \alpha = 1, 2, 3.$$

It follows results that the 3-structure $\{\underset{\alpha}{F}_1\}$ cut off to the distribution $W_1^{\perp}$ agrees with the initial 3-structure $\{\underset{\alpha}{F}\}$ - Theorem 1.3,[2]. The paper [2] (Theorem 1.1) showed that the distributions W_1 and $W_1^{\perp}$ were invariant in relation to $\{\underset{\alpha}{F}^1\}$. Thus, the 3-structure $\{\underset{\alpha}{F}^1\}$ on the distribution $W_1^{\perp}$ satisfies the conditions (1). Moreover, the metric g^1 for the fields $X, Y \in W_1^{\perp}$ satisfies the condition (2), what follows from (8) and (9). The above construction can be repeated for the 3-structure $\{\underset{\alpha}{F}^1\}$ on the distribution $W_1^{\perp}$ which is still denoted by $\tilde{V}_2$, $W_1^{\perp} = \tilde{V}_2$. Then it is made one after the other (totally n times) for the 3-structures obtained from the former ones on corresponding distributions. Thus there is obtained $4n$ vector fields $N_1, \underset{1}{\eta}^1, \underset{2}{\eta}^1, \underset{3}{\eta}^1, \dots, N_n, \underset{1}{\eta}^n, \underset{2}{\eta}^n \underset{3}{\eta}^n$ and $3n$ linear form fields $\underset{1}{\omega}^1, \underset{2}{\omega}^1, \underset{3}{\omega}^1, \dots, \underset{1}{\omega}^n, \underset{2}{\omega}^n, \underset{3}{\omega}^n$ and $3n$ scalar functions $\underset{1}{\lambda}^1, \underset{2}{\lambda}^1, \underset{3}{\lambda}^1, \dots, \underset{1}{\lambda}^n, \underset{2}{\lambda}^n, \underset{3}{\lambda}^n$, whereas for each $i = 1, 2, \dots, n-1$ the following relations take place. Namely, if we assume $\tilde{V}_{i+1} = W_i^{\perp} \subset V_i$ and g^i is the metric on V_i cut off to $\tilde{V}_{i+1}$ and $N_{i+1} \in \tilde{V}_{i+1}$ is any differentiable vector field on M^{4n}, $g^i(N_{i+1}, N_{i+1}) = 1$, $V_{i+1} \subset \tilde{V}_{i+1}$ is the distribution on M^{4n} orthogonal to the field N_{i+1} in relation to the metric, then we obtained

$$\underset{\alpha}{F}^i X = \underset{\alpha}{F}^{i+1} X + \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\omega}^{i+1}(X) N_{i+1} \qquad \text{for } X \in V_{i+1}$$

and

$$\underset{\alpha}{F^i} N_{i+1} = \underset{\alpha}{\eta}^{i+1} + \underset{\alpha}{\varepsilon}\, \underset{\alpha}{\lambda}^{i+1} N_{i+1} \,,$$

where $\underset{\alpha}{F}^{i+1} : \tilde{V}_{i+1} \to V_{i+1}$, $\underset{\alpha}{\omega}^{i+1} : \tilde{V}_{i+1} \to R$, $\underset{\alpha}{\eta}^{i+1} = \underset{\alpha}{F}^{i+1} N_{i+1}$, $\underset{\alpha}{\lambda}^{i+1} = \underset{\alpha}{\omega}^{i+1}(N_{i+1})$. Then for each upper index, the relations (6) and (7) are satisfied. For the metric g^{i+1} induced on V_{i+1} by the condition

$$g^{i+1}(X,Y) = g^{i}(X,Y) \qquad \text{for } X,Y \in V_{i+1}$$

the equalities (8) take place. As before, the decomposition is made

$$\begin{aligned} &\tilde{V}_{i+1} = \{N_{i+1}\} \oplus W_{i+1} \oplus W_{i+1}^{\perp} \,, \\ &V_{i+1} = W_{i+1} \oplus W_{i+1}^{\perp} \,, \\ &\{N_{i+1}\} = \mathrm{Lin}\,(N_{i+1}) \quad , \quad W_{i+1} = \mathrm{Lin}\,(\underset{1}{\eta}^{i+1}, \underset{2}{\eta}^{i+1}, \underset{3}{\eta}^{i+1}) \,. \end{aligned}$$

From now on, it is assumed that each three vectors $\underset{1}{\eta}^{i+1}, \underset{2}{\eta}^{i+1}, \underset{3}{\eta}^{i+1}$ is the linearly independent system ; $W_{i+1}^{\perp}$ is the distribution orthogonal to the distribution W_{i+1} in relation to the metric g^{i+1}. Then $\dim \tilde{V}_{i+1} = 4(n-i+1)$, $\dim V_{i+1} = 4(n-i+1)-1$, $\dim W_{i+1} = 3$, $\dim W_{i+1}^{\perp} = 4(n-i)$.

It follows from (8) that $\underset{\alpha}{\omega}^{i+1}(X) = g^{i+1}(X, \underset{\alpha}{\eta}^{i+1}) = 0$ for $X \in W_{i+1}^{\perp}$.

On the basis of these considerations, the tangent bundle TM^{4n} can be represented as the direct sum :

$$TM^{4n} = \{N_1\} \oplus \cdots \oplus \{N_n\} \oplus W_1 \oplus \cdots \oplus W_n \,,$$

whereas

$$\begin{aligned} &\underset{\alpha}{F}^{k} : W_k \to W_k \,, \qquad k = 1,2,\ldots,n \,, \\ &\underset{\alpha}{F}^{k} \underset{\alpha}{\eta}^{k} = -\underset{\alpha}{\varepsilon}\, \underset{\alpha}{\lambda}^{k} \underset{\alpha}{\eta}^{k} \,, \\ &\underset{\alpha}{F}^{k} \underset{\beta}{\eta}^{k} = \underset{\alpha\beta}{\varepsilon}\, \underset{\gamma}{\eta}^{k} - \underset{\beta}{\varepsilon}\, \underset{\beta}{\lambda}^{k} \underset{\alpha}{\eta}^{k} \end{aligned}$$

(equalities (6) and (7)).

It follows from this construction that for each $\alpha = 1,2,3$ the vector fields $N_1, \ldots, N_n, \underset{\alpha}{\eta}^1, \ldots, \underset{\alpha}{\eta}^n$ are orthogonal each to others and $TM^{4n} = \tilde{V}_1 \supset \tilde{V}_2 \supset \cdots \supset \tilde{V}_n$.

Let us note that 1-forms $\underset{\alpha}{\omega}^{k} : \tilde{V}_k \to R$ satisfy the conditions :

$$\begin{aligned} &\underset{\alpha}{\omega}^{k}(\underset{\alpha}{\eta}^{k}) = 1 - \underset{\alpha}{\varepsilon}(\underset{\alpha}{\lambda}^{k})^2 \\ &\underset{\alpha}{\omega}^{k}(\underset{\beta}{\eta}^{k}) = \underset{\beta\gamma}{\varepsilon}\, \underset{\gamma}{\lambda}^{k} - \underset{\beta}{\varepsilon}\, \underset{\alpha}{\lambda}^{k} \underset{\beta}{\lambda}^{k} \,, \end{aligned}$$

— (6),(7), where $\underset{\alpha}{\lambda}^{k} = \underset{\alpha}{\omega}^{k}(N_k)$ and

$$\underset{\alpha}{\omega}^{k}(X) = 0 \qquad \text{for } X \in W_k^{\perp} \,.$$

1–forms $\underset{\alpha}{\omega}^k$ are extended to the whole tangent bundle $\tilde{V}_1 = TM^{4n}$. Namely, 1–forms

$$\underset{\alpha}{\hat{\omega}}^k : TM^{4n} \longrightarrow R$$

are defined in the following way :

$$\underset{\alpha}{\hat{\omega}}^k(X) = \underset{\alpha}{\omega}^k(X) \quad \text{for } X \in \tilde{V}_k$$
$$\underset{\alpha}{\hat{\omega}}^k(X) = 0 \qquad \text{for } X \in TM^{4n} - \tilde{V}_k \,.$$

This extension is natural as each form $\underset{\alpha}{\omega}^k$ can take values different from zero only on the distribution W_k.

2. On certain distribution of the tangent bundle related to the 3–structure. Now let

$$\underset{\alpha}{U}^n = \mathrm{Lin}\,\{\underset{\alpha}{\eta}^1, \ldots, \underset{\alpha}{\eta}^n\}$$
$$R^n = \{N_1\} \oplus \cdots \oplus \{N_n\} \,.$$

Then

$$TM^{4n} = \underset{1}{U}^n \oplus \underset{2}{U}^n \oplus \underset{3}{U}^n \oplus R^n \,, \tag{10}$$

$\dim \underset{\alpha}{U}^n = n$ and the vectors $\underset{\alpha}{\eta}^1, \ldots, \underset{\alpha}{\eta}^n$ make the orthogonal base of the distribution $\underset{\alpha}{U}^n$.

By means of the mappings $\underset{\alpha}{F}^k$, $\alpha = 1,2,3$, $k = 1,\ldots,n$, there are defined three mappings $\underset{1}{G}, \underset{2}{G}, \underset{3}{G}$ of the direct sum of the distribution $\underset{1}{U}^n \oplus \underset{2}{U}^n \oplus \underset{3}{U}^n$ into $\underset{1}{U}^n \oplus \underset{2}{U}^n \oplus \underset{3}{U}^n$ in such a way that

$$\begin{aligned} &\underset{\alpha}{G} : \underset{\alpha}{U}^n \longrightarrow \underset{\alpha}{U}^n \\ &\underset{\alpha}{G} : \underset{\beta}{U}^n \longrightarrow \underset{\alpha}{U}^n \oplus \underset{\gamma}{U}^n \qquad \alpha \neq \beta \neq \gamma \neq \alpha \,. \end{aligned} \tag{11}$$

These mappings are defined by their values on the vectors $\underset{\alpha}{\eta}^k$ in the following way :

$$\underset{\alpha}{G}\underset{\alpha}{\eta}^k = \underset{\alpha}{F}^k \underset{\alpha}{\eta}^k = -\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k \underset{\alpha}{\eta}^k$$
$$\underset{\alpha}{G}\underset{\beta}{\eta}^k = \underset{\alpha}{F}^k \underset{\beta}{\eta}^k = \underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\eta}^k - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k \underset{\alpha}{\eta}^k$$

Hence, in the base of the vectors $\underset{1}{\eta}^1, \ldots, \underset{1}{\eta}^n, \underset{2}{\eta}^1, \ldots, \underset{2}{\eta}^n, \underset{3}{\eta}^1, \ldots, \underset{3}{\eta}^n$ the matrices of $\underset{1}{G}, \underset{2}{G}, \underset{3}{G}$ are of the corresponding forms

$$B_1 = \begin{bmatrix} -\underset{1}{\varepsilon}\underset{1}{C} & -\underset{2}{\varepsilon}\underset{2}{C} & -\underset{3}{\varepsilon}\underset{3}{C} \\ 0 & 0 & \underset{13}{\varepsilon} I \\ 0 & \underset{12}{\varepsilon} I & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 & \underset{23}{\varepsilon} I \\ -\underset{1}{\varepsilon}\underset{1}{C} & -\underset{2}{\varepsilon}\underset{2}{C} & -\underset{3}{\varepsilon}\underset{3}{C} \\ \underset{21}{\varepsilon} I & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & \underset{32}{\varepsilon} I & 0 \\ \underset{31}{\varepsilon} I & 0 & 0 \\ -\underset{1}{\varepsilon}\underset{1}{C} & -\underset{2}{\varepsilon}\underset{2}{C} & -\underset{3}{\varepsilon}\underset{3}{C} \end{bmatrix}$$

where I – the identity matrix $n \times n$

$$\underset{\alpha}{C} = \begin{bmatrix} \underset{\alpha}{\lambda^1} & & & 0 \\ & \underset{\alpha}{\lambda^2} & & \\ & & \ddots & \\ 0 & & & \underset{\alpha}{\lambda^n} \end{bmatrix} \qquad \alpha = 1,2,3 .$$

For any vector field $\underset{\alpha}{X} \in \underset{\alpha}{U^n}$, $\underset{\alpha}{X} = \sum_{k=1}^{n} \underset{\alpha}{X^k}\, \underset{\alpha}{\eta^k}$ there is then

$$\underset{\alpha}{G}\,\underset{\alpha}{X} = \sum_{k=1}^{n} \underset{\alpha}{X^k}\, \underset{\alpha}{G}\, \underset{\alpha}{\eta^k} = -\underset{\alpha}{\varepsilon} \sum_{k=1}^{n} \underset{\alpha}{X^k}\, \underset{\alpha}{\lambda^k} \underset{\alpha}{\eta^k} ,$$

but for $\underset{\beta}{X} \in \underset{\beta}{U^n}$:

$$\underset{\alpha}{G}\,\underset{\beta}{X} = \sum_{k=1}^{n} \underset{\beta}{X^k}\, \underset{\alpha}{G}\, \underset{\beta}{\eta^k} = \sum_{k=1}^{n} \underset{\beta}{X^k} \big(\underset{\alpha\beta}{\varepsilon}\, \underset{\gamma}{\eta^k} - \underset{\beta}{\varepsilon}\, \underset{\beta}{\lambda^k} \underset{\alpha}{\eta^k} \big) .$$

Hence the conditions (10) are satisfied.

On the distribution $\underset{1}{U^n} \oplus \underset{2}{U^n} \oplus \underset{3}{U^n}$ 1–forms $\underset{\alpha}{\tilde{\omega}^k}$ $\alpha = 1,2,3$, $k = 1,\dots,n$ are also defined in the following way :

$$\underset{\alpha}{\tilde{\omega}^k}(\underset{\alpha}{X}) = \sum_{j=1}^{n} \underset{\alpha}{X^j}\, \underset{\alpha}{\tilde{\omega}^k}(\underset{\alpha}{\eta^j}) = \sum_{j=1}^{n} \underset{\alpha}{X^j}\, \underset{\alpha}{\hat{\omega}^k}(\underset{\alpha}{\eta^j}) = \underset{\alpha}{X^k}\, \underset{\alpha}{\omega^k}(\underset{\alpha}{\eta^k}) = \underset{\alpha}{X^k}\big(1 - \underset{\alpha}{\varepsilon}(\underset{\alpha}{\lambda^k})^2\big)$$

$$\underset{\alpha}{\tilde{\omega}^k}(\underset{\beta}{X}) = \sum_{j=1}^{n} \underset{\beta}{X^j}\, \underset{\alpha}{\tilde{\omega}^k}(\underset{\beta}{\eta^j}) = \sum_{j=1}^{n} \underset{\beta}{X^j}\, \underset{\alpha}{\hat{\omega}^k}(\underset{\beta}{\eta^j}) = \underset{\beta}{X^k}\, \underset{\alpha}{\omega^k}(\underset{\beta}{\eta^k}) = \underset{\beta}{X^k}\big(\underset{\beta\gamma}{\varepsilon}\, \underset{\gamma}{\lambda^k} - \underset{\beta}{\varepsilon}\, \underset{\alpha}{\lambda^k} \underset{\beta}{\lambda^k} \big) .$$

Then as it can be easily checked for the distribution $\underset{\alpha}{U^n}$ the following relations take place :

$$(\underset{\alpha}{G})^2 = \underset{\alpha}{\varepsilon}\big(I - \sum_{k=1}^{n} \underset{\alpha}{\tilde{\omega}^k} \otimes \underset{\alpha}{\eta^k}\big)$$

$$\underset{\alpha}{G} \circ \underset{\beta}{G} = \underset{\alpha\beta}{\varepsilon}\, \underset{\gamma}{G} - \underset{\beta}{\varepsilon} \sum_{k=1}^{n} \underset{\beta}{\tilde{\omega}^k} \otimes \underset{\alpha}{\eta^k}$$

3. Conditions of integrability of $\underset{\alpha}{U^n}$ distribution. We shall find integrability conditions of $\underset{\alpha}{U^n}$ distribution. Any vector $\tilde{X} \in TM^{4n}$ can be written in the form

$$\tilde{X} = \sum_{\beta=1}^{3} \sum_{k=1}^{n} \underset{\beta}{X^k}\, \underset{\beta}{\eta^k} + \sum_{k=1}^{n} \mu^k(\tilde{X}) N_k ,$$

where

$$\mu^k(\bar X) = \bar g(\bar X, N_k)\,.$$

Then

$$\underset{\alpha}{\bar F}\bar X = \sum_{k=1}^{n}[\underset{\alpha}{X}^k \underset{\alpha}{\bar F}\,\underset{\alpha}{\eta}^k + \underset{\beta}{X}^k \underset{\alpha}{\bar F}\,\underset{\beta}{\eta}^k + \underset{\gamma}{X}^k \underset{\alpha}{\bar F}\,\underset{\gamma}{\eta}^k + \mu^k(\bar X)\underset{\alpha}{\bar F}\, N_k] =$$

$$= \sum_{k=1}^{n}[\underset{\alpha}{X}^k(\underset{\alpha}{F}^k\underset{\alpha}{\eta}^k + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega}^k(\underset{\alpha}{\eta}^k)N_k) + \underset{\beta}{X}^k(\underset{\alpha}{F}^k\underset{\beta}{\eta}^k + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega}^k(\underset{\beta}{\eta}^k)N_k) +$$

$$+ \underset{\gamma}{X}^k(\underset{\alpha}{F}^k\underset{\gamma}{\eta}^k + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\omega}^k(\underset{\gamma}{\eta}^k)N_k) + \mu^k(\bar X)(\underset{\alpha}{\eta}^k + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k N_k)] =$$

$$= \sum_{k=1}^{n}[\underset{\alpha}{X}^k(-\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\alpha}{\eta}^k + (\underset{\alpha}{\varepsilon} - (\underset{\alpha}{\lambda}^k)^2)N_k) + \underset{\beta}{X}^k(\underset{\alpha\beta}{\varepsilon}\,\underset{\gamma}{\eta}^k - \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k\underset{\alpha}{\eta}^k + \underset{\alpha}{\varepsilon}(\underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\beta}{\lambda}^k)N_k) +$$

$$+ \underset{\gamma}{X}^k(\underset{\alpha\gamma}{\varepsilon}\,\underset{\beta}{\eta}^k - \underset{\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k\underset{\alpha}{\eta}^k + \underset{\alpha}{\varepsilon}(\underset{\gamma\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k - \underset{\gamma}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\gamma}{\lambda}^k)N_k) + \mu^k(\bar X)(\underset{\alpha}{\eta}^k + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k N_k)]\,.$$

Then

$$\begin{aligned}\underset{\alpha}{\bar F}\bar X = &-\sum_{k=1}^{n}[\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\alpha}{X}^k + \underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k\underset{\beta}{X}^k + \underset{\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k\underset{\gamma}{X}^k - \mu^k(\bar X)]\underset{\alpha}{\eta}^k + \underset{\alpha\gamma}{\varepsilon}\sum_{k=1}^{n}\underset{\gamma}{X}^k\,\underset{\beta}{\eta}^k +\\ &+ \underset{\alpha\beta}{\varepsilon}\sum_{k=1}^{n}\underset{\beta}{X}^k\,\underset{\gamma}{\eta}^k + \sum_{k=1}^{n}[\underset{\alpha}{X}^k(\underset{\alpha}{\varepsilon} - (\underset{\alpha}{\lambda}^k)^2) + \underset{\alpha}{\varepsilon}\,\underset{\beta}{X}^k(\underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k - \underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\beta}{\lambda}^k) +\\ &+ \underset{\alpha}{\varepsilon}\,\underset{\gamma}{X}^k(\underset{\gamma\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k - \underset{\gamma}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\gamma}{\lambda}^k) + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\mu^k(\bar X)]N_k\,.\end{aligned} \tag{12}$$

The Gauss–Codazzi equations for the distribution (10) take the form

$$\bar\nabla_X Y = \nabla_X Y + h(X,Y)$$
$$\bar\nabla_X N = -A_N X + \nabla^{\perp}_X N\,,$$

where $X,\ Y,\ \nabla_X Y,\ A_N X \in \underset{1}{U}^n \oplus \underset{2}{U}^n \oplus \underset{3}{U}^n$, $N,\ h(X,Y),\ \nabla^{\perp}_X N \in R^n =$ $= \{N_1\} \oplus \cdots \oplus \{N_n\}$, $h(X,Y) = h(Y,X)$.

The following notations are introduced

$$\begin{cases} A_{N_i}X = A^i X \\ A^i\underset{\alpha}{\eta}^j = \sum\limits_{\beta=1}^{3}\sum\limits_{k=1}^{n} a^{ijk}\underset{\beta}{\eta}^k \\ \nabla_{\underset{\alpha}{\eta}^j}N^i = \sum\limits_{k=1}^{n}\underset{\alpha}{b}^{ijk}N_k\,. \end{cases} \tag{13}$$

If $\overset{0}{\bar\nabla}$ is the operator of $\{\underset{\alpha}{\bar F}\}$ – connection on the Riemannian manifold M^{4n} ([3]), i.e. $\overset{0}{\bar\nabla}\underset{\alpha}{\bar F} = 0$ for $\alpha = 1, 2, 3$, then for

$$\underset{\alpha}{\bar F}N_i = \underset{\alpha}{\eta}^i + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i N_i$$

there is obtained

$$\underset{\alpha}{\bar F}(\overset{0}{\bar\nabla}_{\underset{\alpha}{\eta}^j} N_i) = \overset{0}{\bar\nabla}_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\eta}^i + \underset{\alpha}{\varepsilon}(\partial_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\lambda}^i)N_i + \underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i\,\overset{0}{\bar\nabla}_{\underset{\alpha}{\eta}^j}N_i$$

$$\underset{\alpha}{\bar F}(-A^i\underset{\alpha}{\eta}^j + \nabla^{\perp}_{\underset{\alpha}{\eta}^j}N_i) = \overset{0}{\nabla}_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\eta}^i + h(\underset{\alpha}{\eta}^j,\underset{\alpha}{\eta}^i) + \underset{\alpha}{\varepsilon}(\partial_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\lambda}^i)N_i + \tag{14}$$

$$+\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i(-A^i\underset{\alpha}{\eta}^j + \overset{0}{\nabla}{}^{\perp}_{\underset{\alpha}{\eta}^j}N_i)\,.$$

Using the above notations and relations and putting in (12)

$$\underset{\beta}{X}^k = -\underset{\alpha\beta}{a}^{ijk} \quad , \quad \mu^k = \underset{\alpha}{b}^{ijk}$$

there is obtained from (14) :

$$\sum_{k=1}^{n}[\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\alpha\alpha}{a}^{ijk}+\underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k\underset{\alpha\beta}{a}^{ijk}+\underset{\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k\underset{\alpha\gamma}{a}^{ijk}+\underset{\alpha}{b}^{ijk}]\underset{\alpha}{\eta}^k-\underset{\alpha\gamma}{\varepsilon}\sum_{k=1}^{n}\underset{\alpha\gamma}{a}^{ijk}\underset{\beta}{\eta}^k-\underset{\alpha\beta}{\varepsilon}\sum_{k=1}^{n}\underset{\alpha\beta}{a}^{ijk}\underset{\gamma}{\eta}^k+$$

$$-\sum_{k=1}^{n}[\underset{\alpha\alpha}{a}^{ijk}(\underset{\alpha}{\varepsilon}-(\underset{\alpha}{\lambda}^k)^2)+\underset{\alpha}{\varepsilon}\,\underset{\alpha\beta}{a}^{ijk}(\underset{\beta\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k-\underset{\beta}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\beta}{\lambda}^k)+\underset{\alpha}{\varepsilon}\,\underset{\alpha\gamma}{a}^{ijk}(\underset{\gamma\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k-\underset{\gamma}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\gamma}{\lambda}^k)-\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\alpha}{b}^{ijk}]N_k=$$

$$=\overset{0}{\nabla}_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\eta}^i+h(\underset{\alpha}{\eta}^j,\underset{\alpha}{\eta}^i)+\underset{\alpha}{\varepsilon}(\partial_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\lambda}^i)N_i-\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i\sum_{k=1}^{n}[\underset{\alpha\alpha}{a}^{ijk}\underset{\alpha}{\eta}^k+\underset{\alpha\beta}{a}^{ijk}\underset{\beta}{\eta}^k+\underset{\alpha\gamma}{a}^{ijk}\underset{\gamma}{\eta}^k]+\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i\sum_{k=1}^{n}\underset{\alpha}{b}^{ijk}N_k$$

Then

$$\overset{0}{\nabla}_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\eta}^i=-h(\underset{\alpha}{\eta}^j,\underset{\alpha}{\eta}^i)+\sum_{k=1}^{n}[\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k\underset{\alpha\alpha}{a}^{ijk}+\underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k\underset{\alpha\beta}{a}^{ijk}+\underset{\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k\underset{\alpha\gamma}{a}^{ijk}+\underset{\alpha}{b}^{ijk}]\underset{\alpha}{\eta}^k+$$

$$-\underset{\alpha\gamma}{\varepsilon}\sum_{k=1}^{n}\underset{\alpha\gamma}{a}^{ijk}\underset{\beta}{\eta}^k-\underset{\alpha\beta}{\varepsilon}\sum_{k=1}^{n}\underset{\alpha\beta}{a}^{ijk}\underset{\gamma}{\eta}^k+\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^i\sum_{k=1}^{n}[\underset{\alpha\alpha}{a}^{ijk}\underset{\alpha}{\eta}^k+\underset{\alpha\beta}{a}^{ijk}\underset{\beta}{\eta}^k+\underset{\alpha\gamma}{a}^{ijk}\underset{\gamma}{\eta}^k].$$

Hence, there is :

$$[\underset{\alpha}{\eta}^j,\underset{\alpha}{\eta}^i]=\overset{0}{\nabla}_{\underset{\alpha}{\eta}^j}\underset{\alpha}{\eta}^i-\overset{0}{\nabla}_{\underset{\alpha}{\eta}^i}\underset{\alpha}{\eta}^j=$$

$$=\sum_{k=1}^{n}[\underset{\alpha}{\varepsilon}\,\underset{\alpha}{\lambda}^k(\underset{\alpha\alpha}{a}^{ijk}-\underset{\alpha\alpha}{a}^{jik})+\underset{\beta}{\varepsilon}\,\underset{\beta}{\lambda}^k(\underset{\alpha\beta}{a}^{ijk}-\underset{\alpha\beta}{a}^{jik})+\underset{\gamma}{\varepsilon}\,\underset{\gamma}{\lambda}^k(\underset{\alpha\gamma}{a}^{ijk}-\underset{\alpha\gamma}{a}^{jik})+(\underset{\alpha}{b}^{ijk}-\underset{\alpha}{b}^{jik})]\underset{\alpha}{\eta}^k+$$

$$+\underset{\alpha}{\varepsilon}\sum_{k=1}^{n}[(\underset{\alpha}{\lambda}^i\underset{\alpha\alpha}{a}^{ijk}-\underset{\alpha}{\lambda}^j\underset{\alpha\alpha}{a}^{jik})\underset{\alpha}{\eta}^k+(\underset{\alpha}{\lambda}^i\underset{\alpha\beta}{a}^{ijk}-\underset{\alpha}{\lambda}^j\underset{\alpha\beta}{a}^{jik})\underset{\beta}{\eta}^k+(\underset{\alpha}{\lambda}^i\underset{\alpha\gamma}{a}^{ijk}-\underset{\alpha}{\lambda}^j\underset{\alpha\gamma}{a}^{jik})\underset{\gamma}{\eta}^k]+$$

$$-\underset{\alpha\beta}{\varepsilon}\sum_{k=1}^{n}(\underset{\alpha\beta}{a}^{ijk}-\underset{\alpha\beta}{a}^{jik})\underset{\gamma}{\eta}^k-\underset{\alpha\gamma}{\varepsilon}\sum_{k=1}^{n}(\underset{\alpha\gamma}{a}^{ijk}-\underset{\alpha\gamma}{a}^{jik})\underset{\beta}{\eta}^k.$$

The necessary and sufficient condition of integrability of the distribution $\underset{\alpha}{U}^n$ there are the following identities

$$(15)\qquad \underset{\alpha\beta}{\varepsilon}\sum_{k=1}^{n}(\underset{\alpha\beta}{a}^{ijk}-\underset{\alpha\beta}{a}^{jik})-\underset{\alpha}{\varepsilon}\sum_{k=1}^{n}(\underset{\alpha}{\lambda}^{i}\underset{\alpha\gamma}{a}^{ijk}-\underset{\alpha}{\lambda}^{j}\underset{\alpha\gamma}{a}^{jik})=0\,.$$

So, the following theorem is obtained :

Theorem . *If* $\overset{0}{\nabla}$ *is the operator of* $\{\underset{\alpha}{F}\}$ *– connection on the Riemannian manifold* M^{4n} *then the necessary and sufficient condition of integrability of the distribution* $\underset{\alpha}{U}^n=\mathrm{Lin}\{\underset{\alpha}{\eta}^1,\dots,\underset{\alpha}{\eta}^n\}$ *is satisfying the equation* (15) *where the coefficients* $\underset{\alpha\beta}{a}^{ijk}$ *are given by* (13).

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STRESZCZENIE

W pracy tej rozważany jest rozkład rozmaitości Riemanna wymiaru $4n$ na iloczyn prosty R^n i trzech n-wymiarowych rozmaitości z zadanymi na nich odpowiednimi strukturami. 3-struktury te generują pewną 3-strukturę na rozmaitości $4n$-wymiarowej. Konstrukcja ta oparta jest na pojęciach pola wektorowego i dystrybucji. Zostały podane również warunki całkowalności tych dystrybucji.