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Properties of Integral Operators Corresponding to Ordinary Differential Equations in Banach Spaces

Własności operatorów całkowych związanych z równaniami różniczkowymi zwyczajnymi w przestrzeniach Banacha

Abstract. Using Darbo's fixed point theorem, it is shown that initial value problems for ordinary differential equations in Banach spaces have a solution provided the right hand side of the equation is a sum of a β-Lipschiz and dissipative function.

Let E be a Banach space with norm $|\cdot|$, $T > 0$, and $f : [0,T] \times E \to E$ a continuous function with $\sup\{|f(t,x)| : 0 \le t \le T,\ X \in E\} \le M$. We denote by $C([0,T],E)$ the Banach space of continuous functions $x : [0,T] \to E$ provided with the norm $\|x\| = \max\{|x(t)| : 0 \le t \le T\}$. The existence of solutions of the initial value problem

$$(*) \qquad x(0) = a\ , \quad x' = f(t,x) \quad (0 \le t \le T)$$

can be proved by finding a fixed point of the operator $\Phi : C([0,T],E) \to C([0,T],E)$ defined by

$$\Phi x(t) = a + \int_0^t f(\tau, x(\tau))\,d\tau \quad (x \in C([0,],E),\ 0 \le t \le T)\ .$$

A function f is called β–Lipsichitz, if it satisfies

$$\beta(f[0,T] \times A)) \le \lambda\, \beta(A) \quad (A \subseteq E \text{ bounded})\ .$$

β denotes the Hausdorf measure of noncompactness (for definition and properties see [2], [8]). A compact function f is β–Lipschitz with $\lambda = 0$. If the function f is Lipschitz with constant λ, i.e.

$$|f(t,x) - f(t,y)| \le \lambda|x - y| \quad (0 \le t \le T\ ;\ x, y \in E)\ ,$$

then it is β-Lipschitz with constant λ. The following results are known.

Theorem 1.

(i) *f is compact $\Longrightarrow$ Φ is compact and $\Phi(C[0,T],E) \subseteq L$.*

(ii) *f is Lipschitz with constant λ $\Longrightarrow$ Φ is Lipschitz with constant λT.*

(iii) *f is β-Lipschitz with constant λ $\Longrightarrow$ $\Phi : L \to L$ is β-Lipschitz with constant λT* [10].

Here L denotes the convex, closed and bounded subset $\{x \in C([0,T],E) : x(0) = a, |x(t) - xs)| \le M|t-s|, 0 \le t, s \le T\}$ of $C([0,T],E)$. In case (i) the fixed point theorem of Schauder gives a fixed point of Φ. If $\lambda T < 1$ we obtain a fixed point of Φ by means of the fixed point theorems of Banach (case (iii) and of Darbo [3] case (iii)).

Martin [6] used approximate solutions to show that the initial value problem $(*)$ has a unique solution, if f is disspative, i.e.

$$[x, y, f(t,x) - f(t,y)]_- \le \lambda |x-y| \quad (0 \le t \le T \; ; \; x, y \in E) \, ,$$

where $[x,y]_- = \lim_{h \downarrow 0} \frac{1}{h}(|x| - |x - hy|)$.

We do not know, if this case can also be treated by a fixed point theorem. As generalization of Theorem 1 we will prove

Theorem 2. *Let $g, k : [0,T] \times E \to E$ be continuous functions with*

$$\sup\{|g(t,x)| : 0 \le t \le T \; , \; x \in E\} + \sup\{|k(t,x)| : 0 \le t \le T \; , \; x \in E\} \le M \tag{1}$$

such that g is disspative with constant λ (> 0) and k is β-Lipschitz with constant κ. Define $\Psi : C([0,T],E) \to C([0,T],E)$ by $\Psi(x) = y$ where y is the unique solution of

$$y(0) = a \; , \quad y' = g(t,y) + k(t, x(t)) \quad (0 \le t \le T) \, .$$

Then Ψ is continuous, $\Psi(C[0,T],E) \subseteq L$ and $\Psi : L \to L$ is β-Lipschitz with constant $q = \frac{\kappa}{\lambda}(e^{\lambda T} - 1)$.

We will use the following lemmata.

Lemma 1 [1]. *Let $\mathcal{A}$ be a bounded and equicontinuous subset of $C([0,T],E)$ and $A = \{x(t) : x \in \mathcal{A}, \; 0 \le t \le T\}$. Then*

$$\beta(A) = \beta(\mathcal{A}) \, .$$

Lemma 2 [9]. *Let $k : [0,T] \times E \to E$ be a continuous, bounded and β-Lipschitz function with constant κ and $A \subseteq E$ bounded. Then to each $\varepsilon > 0$ there exists a finite dimensional subspace Y of E and a continuous and bounded function $s : [0,T] \times E \to Y$ with the property*

$$|k(t,x) - s(t,x)| \le \kappa \beta(A) + \varepsilon \quad (0 \le t \le T, \; x \in A) \, . \tag{2}$$

Proof of Theorem 2. We define $\Gamma : C([0,T],E) \to C([0,T],E)$ by $\Gamma(x) = y+x$, where y is the unique solution of

$$y(0) = a - x(0)\,, \quad y' = g(t, y + x(t)) \quad (0 \le t \le T)\,,$$

or equivalently $\Gamma(x) = z$, where z is the unique solution of

$$z(t) = a + \int_0^t g(\tau, z(\tau))\,d\tau + x(t) \quad (0 \le t \le T)\,. \tag{3}$$

If $K : C([0,T],E) \to C([0,T],E)$ denotes the Nemytskii operator $Kx(t) = k(t, x(t))$ ($x \in C([0,T],E)$, $0 \le t \le T$) and $I : C([0,T],E) \to C([0,T],E)$ the Integral operator $Ix(t) = \int_0^t x(\tau)\,d\tau$ ($x \in C([0,T],E)$, $0 \le t \le T$), we claim

(i) $\Gamma \circ I \circ K = \Psi$, (4)

(ii) Γ is continuous,

(iii) $\Gamma \circ I$ is Lipschitz with constant $\frac{1}{\lambda}(e^{\lambda T} - 1)$. (5)

Equation (4) easily follows from (3) and the definition of Ψ. Since the propositions (ii) and (iii) can be proved by analogous arguments, we only show (ii). Let $x, u \in C([0,T],E)$ and $\Gamma(x) = x + y$, $\Gamma(u) = u + v$. Then $y, v : [0,T] \times E \to E$ are differentiable functions and, with the properties of $[\cdot,\cdot]_-$ (see Lemma II 5.6 and Lemma VI 4.1 in [7]), we get for $t > 0$

$$\begin{aligned}
|y(y) - v(t)|'_- &= [y(t) - v(t), y'(t) - v'(t)]_- \\
&= [y(t) - v(t), g(t, x(t) + y(t) - g(t, u(t) + v(t))]_- \\
&\le [y(t - v(t), g(t, u(t + y(t) - g(t, u(t) + v(t))]_- + |w(t)| \\
&\le \lambda |y(t) - v(t)| + |w(t)|\,,
\end{aligned}$$

where $w(t) = g(t, x(t) + y(t)) - g(t, u(t) + y(t))$ $(0 \le t \le T)$. From the above inequality and $|y(0) - v(0)| = |x(0) - u(0)|$ we deduce by means of well-known theorems on differential inequalities (see Walter [12])

$$\|y - v\| = \|\Gamma x - \Gamma u - (x - u)\| \le e^{\lambda T}|x(0) - u(0)| + \frac{1}{\lambda}(e^{\lambda T} - 1)\|w\|\,.$$

This and the continuity of g gives the continuity of Γ, and with (4) we obtain the continuity of Ψ. Furthermore (1), (3), (4) implies $\Psi(C([0,T],E)) \subseteq L$. Now we will prove that $\Psi : L \to L$ is Lipschitz. Let $\mathcal{A}$ be a subset of L and $A = \{x(t) : x \in \mathcal{A}, 0 \le t \le T\}$. Then by Lemma 1,

$$\beta(A) = \beta(\mathcal{A})\,. \tag{6}$$

We choose $\varepsilon > 0$. By Lemma 2 there exist a finite dimensional subspace Y of E and a continuous and bounded functions $s : [0,T] \times E \to Y$ with property (1). If $S : C([0,T],E) \to C([0,T],E)$ denotes the Nemytskii operator of s, the Arcelà–Ascoli theorem shows the relative compactness of $I \circ S(C([0,T],E))$ and therefore the relative compactness of $\Gamma \circ I \circ S(C([0,T],E))$. From (2), (5), (6) we deduce for $x \in \mathcal{A}$

$$\|\Gamma \circ I(Kx) - \Gamma \circ I(Sx)\| \le \frac{1}{\lambda}(e^{\lambda T} - 1)(\kappa\beta(\mathcal{A}) + \varepsilon)\,.$$

This implies

$$\beta(\{\Gamma \circ I \circ K(x) - \Gamma \circ I \circ S(x) : x \in \mathcal{A}\}) \le \frac{1}{\lambda}(e^{\lambda T} - 1)(\kappa\beta(\mathcal{A}) + \varepsilon) .$$

Since $\beta(\Gamma \circ I \circ S(\mathcal{A})) = 0$ and $\varepsilon > 0$ is arbitrary, we conclude $\beta(\Psi(\mathcal{A})) = \beta(\Gamma \circ I \circ K(\mathcal{A})) \le q\beta(\mathcal{A})$.

Corollary . *Under the assumptions of Theorem 2 the initial value problem* (*) *with* $f = g + h$ *has a solution* $x \in C([0,T], E)$.

Proof. We first choose $t_0 \in (0,T]$ such that $\frac{\kappa}{\lambda}(e^{\lambda t_0} - 1) < 1$. Then from Theorem 2 and the fixed point theorem of Darbo we obtain a solution $x : [0, t_0] \to E$ of (*). This solution can be continued to the whole interval $[0,T]$.

This corollary answers the question in [4], [11] and is proved without use of a fixed point theorem in [9]. For completness we refer to a result of Lemmert [5], who proves the monotonicity of an operator Ψ corresponding to an initial value problem in ordered Banach spaces, where the right hand side is assumed to satisfy some monotonicity conditions.

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STRESZCZENIE

Stosując twierdzenie Darbo o punkcie stałym wykazano, że zadanie początkowe dla równania różniczkowego zwyczajnego w przestrzeni Banacha ma rozwiązanie, o ile tylko prawa strona równania jest sumą funkcji β–Lipschitzowskiej i dyssypatywnej.