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Generalized Neumann-Poincaré Operator and Chord-arc Curves

Uogólniony operator Neumanna-Poincaré'go i krzywe łuk-cięciwa

Abstract. Let Γ be a rectifiable Jordan curve in the finite plane regular in the sense of Ahlfors-David, i.e. AD-regular.

Let L_0^p, $p > 1$, stand for the class of real-valued functions $x(s)$ on Γ such that

$$\int_\Gamma |x(s)|^p \, ds < +\infty \quad \text{and} \quad \int_\Gamma x(s) \, ds = 0 .$$

If the Cauchy singular integral operator C^Γ acting on L_0^p is split into its real and imaginary parts C_1^Γ and C_2^Γ, resp., then the following characterizations of chord-arc curves in the finite plane can be given.

Γ is a chord-arc iff C_2^Γ is a bounded isomorphism of L_0^p for some $p > 1$. Γ is chord-arc iff $-1, 1$ are regular values of the operator C_1^Γ acting on L_0^p for some $p > 1$.

If $p = 2$ and $\|C^T - C^\Gamma\| < 1$, where T is the unit circle, $|\Gamma| = 2\pi$ and $L_0^p = L_0^p(0, 2\pi)$ then Γ is chord-arc. Some further statements concern the case when $\|C_1^\Gamma\| < 1$ and the operator C_1^Γ acts on L_0^2.

1. Introduction. The spectacular achievement of Louis de Branges [2] overshadowed another brilliant result obtained about the same time by Guy David [5]. David was able to give a complete characterization of locally rectifiable curves Γ and exponents p for which the Cauchy singular integral is a bounded linear operator on the space $L^p(\Gamma)$ of complex-valued functions h on Γ that satisfy

$$\int_\Gamma |h(z)|^p |dz| < +\infty .$$

A locally rectifiable curve Γ is called *regular in the sense of Ahlfors-David*, or AD-*regular*, if there exists a positive constant M such that for any disk $D(a; R)$ the arc-length measure $|\Gamma \cap D(a; R)| \le MR$.

The Cauchy singular integral operator C^Γ is defined as

$$(1.1) \qquad (C^\Gamma h)(z_0) = Ch(z_0) = \frac{1}{\pi i} \, P.V. \int_\Gamma \frac{h(z)\, dz}{z - z_0} := $$

$$= \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{\Gamma \setminus \Gamma_\varepsilon} \frac{h(z)\, dz}{z - z_0}, \quad z_0 \in \Gamma ,$$

where Γ_ε is a subarc of Γ of length 2ε bisected by z_0. We drop the usual factor $1/2$ due to reasons evident from what follows.

According to David the operator $h \mapsto C^\Gamma h$ is bounded on a locally rectifiable curve Γ for some $p > 1$, if and only if Γ is AD-regular. Then it is also bounded for all $p > 1$.

This classical problem has a long history going back to Plemelj, Privalov and others. The partial solution for Lipschitz graphs presented by Calderón at the Helsinki Congress [3] was already considered as a major achievement. For more details cf. the excellent survey article of S. Semmes [10].

If Γ is a Jordan curve in the finite plane we may consider, following Guy David [5], ***complementary Hardy spaces*** $H^p(D_k)$, $k = 1, 2$, on complementary domains D_1, D_2 of an AD-regular curve Γ ($0 \in D_1$, $\infty \in D_2$, $p > 1$). For $g \in H^p(D_2)$ we assume the normalization $g(\infty) = 0$. These classes coincide for AD-regular Γ in the finite plane with the familiar classes $E^p(D_k)$, cf. [6].

Any $f \in H^p(D_1)$ has non-tangential limiting values a.e. on Γ and $\int_\Gamma |f(z)|^p |dz| < +\infty$. The same is true for $g \in H^p(D_2)$. Since the functions $f, g \in H^p(D_k)$ can be recovered from their boundary values by the Cauchy integral formula, we may consider $H^p(D_k)$ as subspaces of $L^p(\Gamma)$.

As shown by David, D_1 and D_2 are domains of Smirnov type, i.e. $H^p(D_1)$, $H^p(D_2)$ are L^p-closures of polynomials, or polynomials in z^{-1}, resp. Moreover, any $h \in L^p(\Gamma)$ has a unique decomposition

$$h(\zeta) = f(\zeta) - g(\zeta) \quad ; \quad f \in H^p(D_1) \quad , \ g \in H^p(D_2) . \tag{1.2}$$

This unique decomposition is performed by Plemelj's formulas

$$f(\zeta) = \frac{1}{2}\,[h(\zeta) - Ch(\zeta)] \quad , \quad g(\zeta) = \frac{1}{2}\,[-h(\zeta) + Ch(\zeta)] \quad , \ \zeta \in \Gamma . \tag{1.3}$$

Hence, for Γ being AD-regular and $h \in L^p(\Gamma)$, $p > 1$:

$$h(\zeta) = f(\zeta) - g(\zeta) \quad , \quad Ch(\zeta) = f(\zeta) + g(\zeta) . \tag{1.4}$$

Therefore $h = f$ on Γ holds, iff $g = 0$, i.e.

$$f = Cf \iff f \in H^p(D_1) . \tag{1.5}$$

Similarly

$$g = -Cg \iff g \in H^p(D_2) . \tag{1.6}$$

Moreover, (1.4)–(1.6) imply $CCh = Cf + Cg = f - g = h$, and we obtain an important observation [9]:

$$C^2 = I \quad , \quad C^{-1} = C , \tag{1.7}$$

where I stands for the identity operator. Thus, for any $p > 1$ and any AD-regular Γ C^Γ is an isomorphism of $L^p(\Gamma)$ being an involution.

Remark 1.1. Any $h \in L^p(\Gamma)$, $p > 1$, generates, according to (1.3), a unique pair f, g of functions belonging to complementary H^p–spaces and we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{h(\zeta)\,d\zeta}{\zeta - z} = \begin{cases} f(z)\,, & z \in D_1\,; \\ g(z)\,, & z \in D_2\,. \end{cases} \tag{1.8}$$

2. Generalized Neumann–Poincaré operator C_1^Γ. If Γ is C^3 then the classical Neumann–Poincaré operator $\mathcal{N}$ has the form

$$(\mathcal{N}h)(z) = -\frac{1}{\pi}\int_\Gamma h(\zeta)\frac{\partial}{\partial n_\zeta}\log|\zeta - z|\,ds \quad ;\ z, \zeta \in \Gamma\,. \tag{2.1}$$

The kernel $k(z,\zeta) = -\dfrac{1}{\pi}\dfrac{\partial}{\partial n_\zeta}\log|\zeta - z|$ has in the C^3–case a continuous extension on $\Gamma \times \Gamma$ and so $\mathcal{N}$ is continuous on the space of continuous, real–valued functions h. Due to the identity $-\dfrac{\partial}{\partial n_\zeta}\log|\zeta - z| = \dfrac{\partial}{\partial s}\arg(z - \zeta)$ we may write

$$\begin{aligned}(\mathcal{N}h)(z) &= \frac{1}{\pi}\int_\Gamma h(\zeta)\,\mathrm{Im}\,\frac{\zeta'(s)}{\zeta(s) - z}\,ds = \\ &= \mathrm{Re}\left\{\frac{1}{\pi i}\int_\Gamma \frac{h(\zeta)\,d\zeta}{\zeta - z}\right\}.\end{aligned} \tag{2.2}$$

If the last integral is understood in the sense of principal value and Γ is AD–regular then $\mathcal{N}$ becomes a bounded linear operator on the space $L^p_{\mathbf{R}}(\Gamma) = L^p_{\mathbf{R}}$ of real–valued functions $h \in L^p(\Gamma)$, $p > 1$. If we split the Cauchy operator (1.1) acting on $x \in L^p_{\mathbf{R}}$, $p > 1$, into its real and imaginary parts C_1 and C_2, resp., i.e.

$$C_1 x = C_1^\Gamma x = \frac{1}{2}(Cx + \overline{Cx})\,, \tag{2.3}$$

$$C_2 x = C_2^\Gamma x = \frac{1}{2i}(Cx - \overline{Cx})\,, \tag{2.4}$$

we obtain bounded linear operators on $L^p_{\mathbf{R}}$, $p > 1$, with $C_1^\Gamma = \mathcal{N}$ for Γ in C^3 and continuous x. In what follows we shall drop the superscript Γ in most cases. Therefore C_1 may be called a generalized Neumann–Poincaré operator acting on $L^p_{\mathbf{R}}(\Gamma)$ and bounded for an AD–regular Jordan curve Γ in the finite plane and $p > 1$. If C_1 is bounded for some $p > 1$, it is also bounded for all $p > 1$. In what follows we take for granted these assumptions on Γ and p.

The formula (1.7) implies immediately the following relations resulting from the identity $C = C_1 + iC_2$:

$$C_1^2 - C_2^2 = I\ ,\quad C_1C_2 = -C_2C_1\ . \tag{2.5}$$

We now establish some properties of the operators C_k. To this end we introduce the subspace $L^p_0(\Gamma) = L^p_0$ being the maximal subspace of $L^p_{\mathbf{R}}$ containing no constant functions except for $x = 0$. Thus $L_0^p = P(L_{\mathbf{R}}^p)$ where P is the projection $x \mapsto x - \int_\Gamma x(s)\,ds/L$ with L standing for the length of Γ.

The introduction of the spaces L_0^p enables us to eliminate constant functions from the competition and this can result e.g. in change of norm of the operators C_k. In the case of unbounded AD regular Γ $L_{\mathbf{R}}^p(\Gamma)$ contains no constant functions except for the null function and an analogous procedure is redundant.

Lemma 2.1. *The operator C_2 is bounded on $L_{\mathbf{R}}^p$. It vanishes on constant functions only and maps L_0^p one-to-one onto its subspace*

$$\tilde{L}_0^p(\Gamma) = \tilde{L}_0^p := C_2(L_0^p) \subset L_0^p . \tag{2.6}$$

Proof. It follows from (2.4) that $\|C_2\| \leq \|C\|$ so C_2 is bounded on $L_{\mathbf{R}}^p$. If $x_0 = \text{const}$ then obviously $C_2 x_0 = 0$. Suppose now that $C_2 x_0 = 0$, i.e. $Cx_0 = \overline{Cx_0}$. Then $Cx_0 = y_0 \in L_{\mathbf{R}}^p$ and consequently, $g = \frac{1}{2}(-x_0 + Cx_0) \in H^p(D_2)$, as well as $f = \frac{1}{2}(x_0 + Cx_0) \in H^p(D_1)$, have identically vanishing imaginary parts. Hence $g = 0$, $f = a = \text{const}$ and finally $f - g = x_0 = a$. Thus

$$C_2 x_0 = 0 \Longleftrightarrow x_0 = \text{const}. \tag{2.7}$$

Hence C_2 is 1:1 on L_0^p. Suppose now that $C_2 x_0 = a = \text{const}$ for some $x_0 \in L_{\mathbf{R}}^p$. Then $Cx_0 = C_1 x_0 + iC_2 x_0 = C_1 x_0 + ia$ and consequently $g = \frac{1}{2}(-x_0 + C_1 x_0 + ia) \in H^p(D_2)$ has a constant imaginary part. Thus $\operatorname{Im} g = a = 0$ since $g(\infty) = 0$ and this shows that $C_2(L_{\mathbf{R}}^p)$ does not contain any constant function except for 0. This proves that C_2 is a one-to-one operator on L_0^p and the inclusion (2.6) follows.

The operator C_2 cannot vanish identically on all $L_{\mathbf{R}}^p$ for an AD-regular Γ. Note that otherwise $L_{\mathbf{R}}^p$ would consist only of constant functions which is obviously absurd. For C_1 we have, to the contrary, the following

Remark 2.2. If $\Gamma = T = \{z : |z| = 1\}$ then $C_1^T = 0$ on L_0^p for all $p > 1$. The converse is also true.

Proof. Trigonometric polynomials $x_N = \sum_{n=1}^N (a_n \cos n\theta + b_n \sin \theta) = \sum_{n=1}^N (\alpha_n e^{in\theta} + \overline{\alpha}_n e^{-in\theta})$, $\alpha_n = \frac{1}{2}(a_n - ib_n)$ are dense in L_0^p; the decomposition (1.2) takes the form $x_N = f - g$, where $f = \sum_{n=1}^N \alpha_n e^{in\theta}$, $g = -\sum_{n=1}^N \overline{\alpha}_n e^{-in\theta}$. Hence by (1.4) $C^T x_N = f + g$ is purely imaginary and consequently $C_1^T x_N = 0$ for all x_N and also for all $x \in L_0^p$, $p > 1$.

Suppose now that $C_1 x = 0$ for all $x \in L_0^p$. Then $Cx = -\overline{Cx}$, $x \in L_0^p$, or

$$\frac{1}{\pi i} P.V. \int_\Gamma x(s) \frac{z'(s)\,ds}{z(s) - z(t)} \equiv \frac{1}{\pi i} P.V. \int_\Gamma x(s) \frac{\overline{z'(s)}\,ds}{\overline{z(s)} - \overline{z(t)}} .$$

Hence $\operatorname{Im} \dfrac{z'(s)}{z(s) - z(t)} = 0$, or $\dfrac{d}{ds} \operatorname{Im} \log[z(s) - z(t)] = 0$ for almost all s, t, and consequently $\arg \dfrac{z(s_2) - z(t)}{z(s_1) - z(t)} = \text{const}$ for all t between s_1 and s_2 which can be arbitrary. This means that T is a circle. Note that $s \mapsto z(s)$ is absolutely continuous for a bounded AD-regular Γ.

3. Neumann domains and Lavrentiev curves. A bounded, simply connected domain D is said to be a *Caratheodory domain*, if the conformal mapping of D on the unit disk Δ has a continuous extension to its closure $\overline{D}$. Any bounded Jordan domain is a Carathéodory domain, however, there exist Carathéodory domains whose boundary is not a Jordan curve.

Let D be a bounded Carathéodory domain with a rectifiable boundary curve Γ and let $H_0^p(D)$ stand for $\{f \in H^p(D) : \operatorname{Re} f \in L_0^p \,,\ \operatorname{Im} f \in L_0^p\}$. According to M. Zinsmeister [11] *D is said to be a Neumann domain iff there exists a bounded linear operator $S : L_0^p \to L_0^p$ generating an isomorphism $\varphi \mapsto \varphi + iS\varphi$ between $L_0^p = L_0^p(\Gamma)$ and $H_0^p(D)$ for some $p > 1$.*

If such an isomorphism does exist for some $p > 1$, it also exists for all $p > 1$. An unbounded domain D_0 with a locally rectifiable boundary curve Γ is called a Neumann domain iff for some $z_0 \notin \overline{D}_0$ the image domain D of D_0 under the mapping $z \mapsto (z - z_0)^{-1}$ is a bounded Neumann domain.

Zinsmeister also gave the following geometric characterization of Neumann domains: *D is a Neumann domain if and only if ∂D is AD-regular and $\mathbf{C} \setminus D$ is k-locally connected for some $k \geq 1$.*

We recall that a set E in the plane is called k-locally connected (k-l.c.), $k \geq 1$, if for any $z_0 \in \mathbf{C}$ and any disk $D(z_0; r)$ the set $E \cap \overline{D}(z_0; r)$ is contained in a connected subset of $E \cap \overline{D}(z_0; kr)$.

Since AD-regularity and k-l.c. property are preserved under Moebius transformations, the above geometric characterization applies to both bounded and unbounded domains.

A Jordan curve Γ in $\overline{\mathbf{C}}$ is called a chord-arc (or Lavrentiev) *curve* iff there exists a positive constant K such that for any $z_1, z_2 \in \Gamma$ we have $\min\{|\Gamma_1|, |\Gamma_2|\} \leq K|z_1 - z_2|$, where Γ_k are complementary subarcs of Γ with end-points z_1, z_2 and length $|\Gamma_k|$. Evidently any chord-arc curve is AD-regular.

As pointed out by Zinsmeister [11], Γ is chord-arc if and only if both components of $\mathbf{C} \setminus \Gamma$ are Neumann domains. In view of Gehring's characterization of quasicircles in terms of k-local connectivity [8] chord-arc curves may be also characterized as AD-regular quasicircles, cf. [11]. Using the above given definitions and characterizations we shall derive various characterizations of chord-arc curves in terms of operators C_k, $k = 1, 2$.

Theorem 3.1. *An AD-regular curve Γ in the finite plane is chord-arc if and only if*

$$\widetilde{L}_0^p(\Gamma) := C_2(L_0^p) = L_0^p \,.$$

Proof. Suppose that $\widetilde{L}_0^p(\Gamma) = L_0^p$. As shown in Lemma 2.1, C_2 maps L_0^p onto $\widetilde{L}_0^p$ one-to-one. Since L_0^p is a Banach space, the inverse mapping C_2^{-1} is a bounded operator on L_0^p and maps it onto itself.

The function $\varphi + i\psi$ with $\varphi, \psi \in L_0^p$ belongs to $H_0^p(D_1)$ if and only if $C(\varphi + i\psi) = C_1\varphi - C_2\psi + i(C_1\psi + C_2\varphi) = \varphi + i\psi$, i.e.

$$-\varphi + C_1\varphi = C_2\psi \quad , \quad -\psi + C_1\psi = -C_2\varphi \,. \tag{3.1}$$

Thus the desired isomorphism between L_0^p and $H_0^p(D_1)$ takes the form $\varphi \mapsto \varphi + iS_1\varphi$, where

$$(3.2) \qquad S_1 = -C_2^{-1}(I - C_1) \quad , \quad S_1^{-1} = -S_1 \, .$$

This proves that D_1 is a Neumann domain. Similarly $\chi + i\psi$ with $\chi, \psi \in L_0^p$ belongs to $H_0^p(D_2) = H^p(D_2)$ if and only if $C(\chi + i\psi) = C_1\chi - C_2\psi + i(C_1\psi + C_2\chi) = -\chi - i\psi$, i.e.

$$(3.3) \qquad \chi + C_1\chi = C_2\psi \quad , \quad \psi + C_1\psi = -C_2\chi$$

and the desired isomorphism between L_0^p and $H^p(D_2)$ takes the form $\chi \mapsto \chi + iS_2\chi$, where

$$(3.4) \qquad S_2 = C_2^{-1}(I + C_1) \quad , \quad S_2^{-1} = -S_2 \, .$$

Thus D_2 is also a Neumann domain and consequently, Γ is chord–arc.

Suppose now that Γ is chord–arc. Then D_1, D_2 are Neumann domains and consequently, there exists bounded linear operators S_1, S_2 such that for an arbitrary $\psi \in L_0^p$

$$\psi + iS_1\psi \in H_0^p(D_1) \quad , \quad \psi + iS_2\psi \in H^p(D_2) \, .$$

Thus $f = -S_1\psi + i\psi \in H_0^p(D_1)$ and $g = -S_2\psi + i\psi \in H^p(D_2)$ have equal imaginary parts and are generated, due to (1.3), (1.4), by $x_0 = f - g = (S_2 - S_1)\psi = 2C_2^{-1}\psi \in L_0^p$. Moreover, $\frac{1}{2}\, C_2x_0 = \psi$ may be arbitrary which shows that $C_2(L_0^p) = L_0^p$ and this ends the proof.

Remark 3.2. *Γ is a chord-arc curve iff S_1, S_2 are bounded on L_0^p for some $p > 1$.*

It follows from the formulas (3.1) that the operator $S_1 : \varphi \mapsto \psi$ may be also defined as the unique solution ψ of the equation $(I - C_1)\psi = C_2\varphi$. Thus there exists a bounded inverse $(I - C_1)^{-1}$ and hence we obtain, in the case of chord–arc curves,

$$(3.5) \qquad S_1 = (I - C_1)^{-1}C_2 = -C_2^{-1}(I - C_1) \, ,$$

and similarly

$$(3.6) \qquad S_2 = -(I + C_1)^{-1}C_2 = C_2^{-1}(I + C_1) \, .$$

Moreover,

$$(3.7) \qquad S_2 - S_1 = 2C_2^{-1}$$

shows to be an isomorphism of L_0^p for a chord–arc curve. Consequently, we obtain the following

Theorem 3.3. *An AD-regular curve $\Gamma \neq T$ in the finite plane is chord-arc if and only if the points $1, -1$ are regular values of the operator C_1^Γ acting on L_0^p, $p > 1$.*

We exclude the case $\Gamma = T$ since then $C_1 = 0$ and so the notion of regular values does not make sense.

As an immediate consequence we obtain

Proposition 3.4. *If Γ is AD-regular and the norm of C_1^Γ w.r.t. L_0^p is equal $d < 1$ for some $p > 1$ then Γ is a chord-arc curve.*

Proof. Given an arbitrary $y \in L_0^p$ we can write the unique solution x of the equation $y = (I - C_1)x$ in the form of an absolutely convergent series $x = y + C_1 y + C_1^2 y + \cdots$ Similarly $x = y - C_1 y + C_1^2 y - \cdots$ is the unique solution of the equation $y = (I + C_1)x$. Moreover,

$$\|(I \mp C_1)x\| \geq (1-d)\|x\|\ , \quad \text{i.e. } \|x\| \leq (1-d)^{-1}\|y\|$$

in both cases. Consequently, ∓ 1 are regular points of the operator C_1 and we are done, in view of Theorem 3.3.

Note that $C_1 x = x$ for $x = \text{const}$ so that $\|C_1\| \geq 1$ on $L_{\mathbf{R}}^p$ and therefore the elimination of constant functions is essential.

In the case $p = 2$ we shall obtain another related sufficient condition for Γ to be chord-arc. Since the norm of C^Γ does not change under similarity, we may assume that $|\Gamma| = 2\pi$. Then both operators C^T, C^Γ act on $L^2(0, 2\pi)$ and the operator $C^T - C^\Gamma$ makes sense. As a simple consequence of Proposition 3.4 and Remark 2.2 we obtain

Proposition 3.5. *If the L_0^p-norm $\|C^T - C^\Gamma\| = d < 1$ then Γ is a chord-arc curve.*

Proof. We have for an arbitrary $x_0 \in L_0^2$ in view of Remark 2.2

$$d^2\|x_0\|^2 \geq \|(C^T - C^\Gamma)x_0\|^2 = \| - C_1^\Gamma x_0 + i(C_2^T - C_2^\Gamma)x_0\|^2 =$$
$$= \|C_1^\Gamma x_0\|^2 + \|(C_2^T - C_2^\Gamma)x_0\|^2$$

Hence $\|C_1^\Gamma x_0\| \leq d\|x_0\|$ and this ends the proof.

Proposition 3.5 is a counterpart of a theorem due to Coifman and Meyer [4] which refers to the unbounded chord-arc curves. Note that for unbounded Γ the space $L_{\mathbf{R}}^p$ does not contain constant functions $\neq 0$.

4. The case $\|C_1^\Gamma\|_{L_0^2} < 1$. Chord-arc curves for which the L_0^2-norm of the generalized Neumann–Poincaré operator C_1 is less than one make up a rather interesting class of curves. First of all, the Neumann series is convergent in L_0^2. Since the Neumann operator C_1 may be written in the form

$$(C_1 x)(t) = \frac{1}{\pi}\ P.V. \int_\Gamma x(s)\, ds \arg(z(s) - z(t))\ ,$$

the condition $\|C_1\|_{L_0^2} < 1$ indicates that the local rotation of the chord emanating from $z(t)$ is fairy small in the mean. We shall now derive some equivalent analytic conditions included in the

Theorem 4.1. *For a chord-arc curve $\Gamma \neq T$ the following are equivalent:*
(i) $\|C_1^{\Gamma}\|_{L_0^2} =: \|C_1\| = d < 1$;
(ii) *for any pair $f = \varphi + i\psi \in H_0^2(D_1)$, $g = \chi + i\psi \in H^2(D_2)$, $\psi \neq 0$, the inner product $\langle\varphi, \chi\rangle$ is negative and*

$$\langle\varphi, \chi\rangle \le -\frac{1}{4}(1-d^2)\|\varphi - \chi\|^2 ;$$

(iii) *the operator $(I + C_1)(I - C_1)^{-1}$ is positive and*

$$\langle (I + C_1)(I - C_1)^{-1} x_0, x_0\rangle \ge \frac{1-d}{1+d}\|x_0\|^2, \quad 0 < d < 1,$$

for any $x_0 \in L_0^2$.

Proof. (i)$\Longleftrightarrow$(ii) According to (1.3) the functions f, g are generated by $x_0 = \varphi - \chi \in L_0^2$. Obviously (i) is equivalent to

$$\|x_0\|^2 - \|C_1 x_0\|^2 \ge (1-d^2)\|x_0\|^2 . \tag{4.1}$$

On the other hand,

$$\varphi = \frac{1}{2}(x_0 + C_1 x_0) \quad , \quad \chi = \frac{1}{2}(-x_0 + C_1 x_0),$$

and hence

$$\langle\varphi, \chi\rangle = \frac{1}{4}(-\|x_0\|^2 + \|C_1 x_0\|^2),$$

or

$$\|x_0\|^2 - \|C_1 x_0\|^2 = -4\langle\varphi, \chi\rangle . \tag{4.2}$$

From (4.1) and (4.2) the equivalence of (i) and (ii) readily follows.

(ii)$\Longrightarrow$(iii). Since Γ is chord-arc, $I - C_1$ is an isomorphism of L_0^2 by Theorem 3.3 and so given $y_0 \in L_0^2$ we can find a unique x_0 satisfying $(I - C_1)^{-1} y_0 = x_0$. Then, as before,

$$\langle (I + C_1)(I - C_1)^{-1} y_0, y_0\rangle = \langle (I + C_1)x_0, (I - C_1)x_0\rangle =$$
$$= \langle 2\varphi, -2\chi\rangle = -4\langle\varphi, \chi\rangle \ge (1-d^2)\|x_0\|^2$$

in view of (ii). Now, $y_0 = (I - C_1)x_0$ and hence $\|y_0\| \le (1+d)\|x_0\|$ by (i)$\Longleftrightarrow$(ii). Thus $\|x_0\|^2 \ge \|y_0\|^2/(1+d^2)$ and finally

$$\langle (I + C_1)(I - C_1)^{-1} y_0, y_0\rangle \ge \frac{1-d^2}{(1+d)^2}\|y_0\|^2 = \frac{1-d}{1+d}\|y_0\|^2 .$$

(iii)$\Longrightarrow$(i). Suppose Γ is chord–arc and

$$(4.3)\qquad \langle (I+C_1)(I-C_1)^{-1}y_0, y_0\rangle \geq \frac{1-d}{1+d}\|y_0\|^2$$

for some $0 < d < 1$ and all $y_0 \in L_0^2$. With $(I-C_1)^{-1}y_0 = x_0$ we have $\|y_0\| \geq \delta\|x_0\|$ for some $\delta > 0$ and all $x_0 \in L_0^2$ so that

$$\langle (I+C_1)x_0, (I-C_1)x_0\rangle = \|x_0\|^2 - \|C_1x_0\|^2 \geq \frac{1-d}{1+d}\delta^2\|x_0\|^2$$

for all $x_0 \in L_0^2$. This implies $\|C_1\| = d_1 < 1$ which is equivalent to (i) with $d = d_1$. Repeating the steps (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) we see that the best value of d in (4.3) is just d_1. This ends the proof.

If $f = \varphi + i\psi \in H_0^2(D_1)$, $g = \chi + i\psi \in H^2(D_2)$ then $\varphi = -S_1S_2\chi$, $\chi = -S_2S_1\varphi$ and in view of (ii) we obtain

Corollary 4.2. *If $\|C_1\| = d < 1$ on L_0^2 then*

$$(4.4)\qquad \langle S_1S_2\chi, \chi\rangle = \langle S_2S_1\varphi, \varphi\rangle \geq \frac{1}{4}(1-d^2)\|\varphi - \chi\|^2$$

for any $\varphi, \chi \in L_0^2$. Thus S_1S_2 and $(S_1S_2)^{-1} = S_2S_1$ are positive.

Corollary 4.3. *If Γ is chord–arc then $H_0^p(D_1)$ and $H^p(D_2)$ are isomorphic. The isomorphism can be established by the formula*

$$f = -S_1\psi + i\psi \Longleftrightarrow g = -S_2\psi + i\psi\,, \quad \psi \in L_0^p.$$

The converse also holds if S_1, S_2 are bounded, due to Zinsmeister's characterization of chord–arc curves.

A natural question arises to find a geometric characterization of curves for which $\|C_1\| < 1$.

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STRESZCZENIE

Niech Γ będzie prostowalną krzywą Jordana w płaszczyźnie skończonej, regularną w sensie Ahlforsa-Davida, tzn. AD-regularną.

Niech L_0^p, $p > 1$, oznacza klasę funkcji rzeczywistych $x(s)$ na Γ takich, że $\int_\Gamma |x(s)|^p ds < +\infty$ oraz $\int_\Gamma x(s)\, ds = 0$. Jeżeli operator całki osobliwej Cauchy'ego C^Γ działający na L_0^p rozłożymy na jego część rzeczywistą C_1^Γ i urojoną C_2^Γ, to można scharakteryzować krzywe łuk-cięciwa w terminach tych operatorów.

Γ jest krzywą łuk-cięciwa wtedy i tylko wtedy, gdy C_2^Γ jest ograniczonym izomorfizmem L_0^p dla pewnego $p > 1$.

Γ jest krzywą łuk-cięciwa wtedy i tylko wtedy, gdy -1, 1 są wartościami regularnymi operatora C_1^Γ działającego na L_0^p dla pewnego $p > 1$.

Jeśli $p = 2$ oraz $\|C^T - C^\Gamma\| < 1$, gdzie T jest okręgiem jednostkowym, $|\Gamma| = 2\pi$ oraz $L_0^p = L_0^p(0, 2\pi)$, to Γ jest krzywą łuk-cięciwa. Ponadto podano kilka dalszych własności operatora C_1^Γ działającego na L_0^2 w przypadku gdy $\|C_1^\Gamma\| < 1$.