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Ring Homomorphisms on Algebras of Analytic Functions

Homomorfizm pierścieniowy algebr funkcji analitycznych

Abstract. Let $H(G)$ and $H(\Gamma)$ be algebras of analytic functions on regions G and Γ, respectively, in the complex plane. It is shown that a ring homomorphism from $H(G)$ into $H(\Gamma)$ is either linear or conjugate linear, provided the ring homomorphism takes the identity function into a nonconstant function. As a consequence, an alternative proof of Ber's theorem is given and this theorem is extended to the several variables case.

Introduction. An operator M from a comutative algebra A into a comutative algebra B is called a ring homomorphism if for all $x, y \in A$, $M(x+y) = M(x) + M(y)$ and $M(xy) = M(x)M(y)$. A ring isomorphism is a ring homomorphism which is one-one and onto. Throughout this paper G and Γ denote regions, i.e., connected open sets in the complex plane. If G is a region then $H(G)$ denotes the algebra of analytic functions on G equipped with the topology of uniform convergence on compact subsets of G, I_G denotes the identity function on G, and M denotes a non-zero ring homomorphism from $H(G)$ into $H(\Gamma)$. The rationals, reals, and complex numbers are denoted by Q, R, and C, respectively.

If $\mathcal{M}$ is a maximal ideal in $H(G)$ then the quotient algebra $H(G)/\mathcal{M}$ is isomorphic (as an algebra) to C if and only if $\mathcal{M}$ is a closed maximal ideal. Henriksen [4] has shown that if the maximal ideal $\mathcal{M}$ in E is not closed, then $E/\mathcal{M}$ is isomorphic (as a ring) to C, where E is the ring of entire functions. This implies that there exist discontinuous homomorphisms from the ring of entire functions onto C.

Bers [2, 6] has shown that $H(G)$ and $H(\Gamma)$ are ring isomorphic if and only if G and Γ are either conformally or anticonformally equivalent. Further he has shown that every ring isomorphism from $H(G)$ onto $H(\Gamma)$ is induced by either a conformal or an anticonformal map. Rudin [10] has similar results on rings of bounded analytic functions. Becker and Zame [1] have shown that a ring homomorphism M from an F-algebra into an analytic ring is linear (or conjugate linear) and continuous, if the range of M contains a nonunit, nonzero divisor. In [3], Burckel and Saeki have characterized additive maps between rings of holomorphic functions which satisfy a multiplier-like condition. In this paper we show that if G and Γ are regions in C

and a ring homomorphism M from $H(G)$ into $H(\Gamma)$ takes the identity function I_G to a non-constant function, then M is necessarily either linear or conjugate linear. A similar result has been proved by the author [8] for ring homomorphisms from $H(G)$ into iself when G is a regular region. Essentially, to achieve this result we show that the homomorphism under consideration preserve constants or take constants to their conjugates. We give a new proof of Ber's theorem (see Bers [2]) based on this result. Finally we extend Ber's theorem to alebras of analytic functions in several complex variables.

If M is a ring homomorphism from $H(G)$ into $H(\Gamma)$ then the following assertions are equivalent:

1) M is continuous,
2) either $M(k) = k$ for all $k \in C$ or $M(k) = \bar{k}$ for all $k \in C$,
3) M is either linear or conjugate linear,
4) there exists $h \in H(\Gamma)$ with $\overline{h(\Gamma)} \subset G$ such that $M(f) = \overline{f \circ h}$ for all $f \in H(G)$ or there exists $h \in H(\Gamma)$ with $\overline{h(\Gamma)} \subset G$ such that $M(f) = f \circ \bar{h}$ for all $f \in H(G)$.

The implications 4) $\Longrightarrow$ 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) are trivial or easy to prove; 3) $\Longrightarrow$ 4) is the content of Lemma 1.

To show that a ring homomorphism M from $H(G)$ into $H(\Gamma)$ which takes the identity function to a non-constant function is necessarily linear or conjugate linear we use Nienhuys–Thiemann's theorem [9] which states that given any two countable dense subsets A and B of R there exists an entire function which is real valued and increasing on the real line R such that $f(A) = B$. In Section 2 we give some lemmas and state the theorem of Nienhuys and Thiemann. In Section 3 we prove the following main result and finally Ber's Theorem is proved in Section 4.

Theorem. *Let G and Γ be regions in C and let M be a ring homomorphism from $H(G)$ into $H(\Gamma)$ such that $M(I_G)$ is not a constant function where I_G is the identity function on G. Then $M(i) = \pm i$. Further*

a) if $M(i) = i$ then M is linear;
b) if $M(i) = -i$ then M is conjugate linear.

2. Lemmas. The following lemma is well known but we give the proof for the sake of completeness.

Lemma 1. *Let M be a ring homomorphism from $H(G)$ into $H(\Gamma)$. If M is linear then there exists an $h \in H(\Gamma)$ with $h(\Gamma) \subset G$ such that $M(f) = f \circ h$ for all $f \in H(G)$.*

Proof. Let $M(I_G) = h$ and $z_0 \in \Gamma$. We claim that $h(z_0) \in G$. Suppose not, then $I_G - h(z_0)$ is invertible in $H(G)$ and

$$(I_G - h(z_0))\left(\frac{1}{I_g - h(z_0)}\right) = 1.$$

Applying M on both sides and evaluating at z_0 with the observation that $M(h(z_0)) =$

$h(z_0)$ we obtain

$$\begin{aligned} 0 &= (M(I_G)(z_0) - h(z_0))M\Big(\frac{1}{I_G - h(z_0)}\Big)(z_0) \\ &= M(I_G - h(z_0))(z_0)M\Big(\frac{1}{I_G - h(z_0)}\Big)(z_0) \\ &= M(1)(z_0) \\ &= 1 \end{aligned}$$

which is a contradiction. Since z_0 is arbitrary we have $h(\Gamma) \subset G$.

Since $h(z_0) \in G$, we have $\dfrac{f - f(h(z_0))}{I_G - h(z_0)} \in H(G)$ for all $f \in H(G)$ and

$$f - f(h(z_0)) = (I_G - h(z_0))\Big(\frac{f - f(h(z_0))}{I_G - h(z_0)}\Big).$$

Applying M on both sides and evaluating at z_0 we obtain

$$M(f)(z_0) = M(f(h(z_0)))(z_0) = f(h(z_0)) \quad \text{for all } f \in H(G).$$

Since z_0 is arbitrary the result follows.

Lemma 2. *Let G and Γ be two regions in C and M be a ring homomorphism from $H(G)$ into $H(\Gamma)$ with $M(i) = i$. If $M(I_G) = h$ is not a constant function then $h(\Gamma) \cap G$ is not empty.*

Proof. Since M is nontrivial ring homomorphism it is easy to show that $M(\alpha) = \alpha$ for all $\alpha \in Q$. Since $M(i) = i$ we have $M(\alpha + i\beta) = \alpha + i\beta$ for $\alpha, \beta \in Q$. Since h is a nonconstant analytic function it is an open map, so there exists a $z_0 \in \Gamma$ such that $h(z_0) \in Q + iQ$. Just as in the above lemma it is easy to show that $h(z_0) \in G$. Hence $h(\Gamma) \cap G$ is not empty.

Let $k \in Q$. Denote by H_k the set of all entire functions which map $Q + ik$ into Q except possibly for one point of $Q + ik$ and also denote by EM the class of entire functions whose restriction to R is a real monotonically increasing function. The proof of Lemma 3 follows the proof of the following theorem [9].

Theorem (Nienhuys, Thiemann). *Let S and T be countable everywhere dense subsets of R. Suppose that p is a continuous positive real function such that $\lim_{t\to\infty} t^{-n}p(t) = \infty$ for all $n \in N$ and suppose $f_0 \in EM$. Then there exists a function $f \in EM$ such that*

i) f is strictly increasing on R and $f(S) = T$,

ii) $|f(z) - f(z_0)| \le p(|z|)$ for all $z \in C$.

Lemma 3. *Let $k \in Q$, $\beta \in R$ and $\alpha \in Q + ik$. Then there exists an entire function $f \in H_k$ such that $f(\alpha) = \beta$ and $f(Q + ik) = \{\beta\} \cup Q$.*

Proof. In Nienhuys and Thiemann's Theorem [9] take $S = Q$ and $T = \{\beta\} \cup Q$. Let $x_1, x_2, \ldots$ be an enumeration of Q with $x_1 = \alpha - ik$. Then as in the proof of that theorem there exists an entire function g such that $g(x_1) = \beta$ and $g(Q) = \{\beta\} \cup Q$. Let $h(z) = z - ik$. Then $f = g \circ h$ is the desired function.

3. Proof of the main theorem. It is easy to see that M is linear over the field of rational numbers and hence we have $-1 = M(-1) = M(i^2) = M(i)^2$, which implies $M(i) = i$ or $M(i) = -i$. We prove here only Part a) of the theorem; the proof of Part b) follows similarly. So in what follows we are assuming $M(i) = i$.

Since $h = M(I_G)$ is a nonconstant analytic function on Γ, $h(\Gamma)$ is a nonempty open set in C and by Lemma 2, $h(\Gamma) \cap G$ is not empty. Hence there exists $k \in Q$ such that $S = (R + ik) \cap h(\Gamma) \cap G$ contains a non-void interval parallel to the real axis. Let $f \in H(G)$ and $h(z_0) \in (Q + ik) \cap G$. Then applying M on both sides and evaluating at z_0 in the following

$$f - f(h(z_0)) = (I_G - h(z_0))\left(\frac{f - f(h(z_0))}{I_G - h(z_0)}\right),$$

we obtain

$$M(f - f(h(z_0)))(z_0) = 0$$

for all z_0 in Γ such that $h(z_0) \in (Q + ik) \cap G$. Thus for all $f \in H(G)$ we have

$$\text{(1)} \qquad M(f)(z_0) = M(f(h(z_0)))(z_0)\,, \quad \text{for all } z_0 \text{ such that } h(z_0) \in (Q + ik) \cap G\,.$$

Since a function f in H_k takes $Q+ik$ into the rationals except for one point of $Q+i$, we obtain $M(f(h(z_0))) = f(h(z_0))$ whenever $h(z_0) \in (Q+ik) \cap G$ except possibly for one point and $f \in H_k$. Since f, h and $M(f)$ are analytic and since $f(h(z_0)) = M(f)(z_0)$ holds for all z_0 in the infinite set $h^{-1}(G \cap (Q + ik))$ we obtain

$$\text{(2)} \qquad M(f) = f \circ h\,, \quad \text{for all } f \in H_k\,.$$

For a given $\beta \in R$ and a given $h(z_0)$ in $Q + ik$, by Lemma 3 there exists an entire f in H_k such that $f(h(z_0)) = \beta$. Substituting this in (1) on the one hand we obtain

$$M(f)(z_0) = M(\beta)(z_0)$$

and evaluating (2) at z_0 on the other hand we find

$$M(f)(z_0) = (f \circ h)(z_0) = f(h(z_0)) = \beta\,.$$

Thus we obtain from the above two relations that

$$M(\beta)(z_0) = \beta \quad \text{for all } z_0 \in h^{-1}(Q + ik) \cap \Gamma\,.$$

Since $M(\beta)$ is analytic we have $M(\beta) = \beta$. Thus we have $M(\zeta) = \zeta$ for all $\zeta \in R$ and thus for all $\zeta \in C$. This implies M is linear.

4. Ber's Theorem.

Theorem. *Let $H(G)$ and $H(\Gamma)$ be algebras of analytic functions on G and Γ, respectively. Let π be a ring isomorphism from $H(G)$ onto $H(\Gamma)$. Then there exists $\varphi \in H(\Gamma)$ such that either φ is either conformal or anticonformal from Γ onto G and*
a) $\pi(f) = f \circ \varphi$, for all $f \in H()$, or
b) $\pi(f) = \overline{f \circ \bar{\varphi}}$, for all $f \in H(G)$.

Proof. Since $\pi(i) = \pm i$, we will only consider the case $\pi(i) = i$; the case $\pi(i) = -i$, follows similarly. Let $\pi(I_G) = \varphi$. We claim that this φ is the required function. It is enough to show that φ is a nonconstant function and is one–one from Γ onto G.

φ is not a constant function. Since isomorphisms take constant functions to constant functions, so do inverse isomorphisms. Hence $\pi(I_G) = \varphi$ is not a constant function.

φ is onto. Since φ is a nonconstant function, by our theorem π is linear and thus by Lemma 1 we have $\varphi(\Gamma) \subseteq G$. Suppose φ is not onto, then there exists $z_0 \in G\backslash\varphi(\Gamma)$. Then $\varphi - z_0 \in H(\Gamma)$ is invertible and $\pi^{-1}(\varphi - z_0) = \pi^{-1}(\varphi) - \pi^{-1}(z_0) = I_G - z_0$ is not invertible. But non-zero homomorphisms take invertible elements to invertible elements. Contradiction.

φ is one–one. Let $\pi^{-1}(I_\Gamma) = \psi$. Since π^{-1} is an isomorphism and ψ is not a conctant, by our theorem we have

$$\pi^{-1}(f) = f \circ \psi\,, \quad \text{for all } f \in H(\Gamma)\,.$$

Thus we have

$$I_G = \pi^{-1}(\pi(I_G)) = \pi^{-1}(\varphi) = \varphi \circ \psi$$

and

$$I_\Gamma = \pi(\pi^{-1}(I_\Gamma)) = \pi(\psi) = \psi \circ \varphi\,,$$

which imply φ is one–one.

5. Bers' Theorem in C^n. In this section we extend Bers' Theorem to several complex variables. We use Michael's theorem (see [7]) regarding multiplicative linear functionals on multiplicatively convex algebras. We primarily use the notation as given in Krantz [5]. We denote by C^n the Cartesian product of n copies of the complex numbers. An element in C^n is denoted by $z = (z_1, z_2, \ldots, z_n)$. If G is a domain in C^n, then $H(G)$ denotes the algebra of analytic functions on G. Let I_j^G in $H(G)$ denote the j^{th} coordinate function on G, i.e., $I_j^G(z) = z_j$ for all $z \in G$.

We denote by M, a ring homommorphism from $H(G)$ into $H(\Gamma)$, where G and Γ are regions in C^n. Since $M(i) = \pm i$, we prove Bers' theorem for the case $M(i) = i$ and the other case follows similarly. For simplicity we assume $n = 2$; for general n the proof is similar.

Theorem. *Let G and Γ be domains of holomorphy in C^2. Let M be a ring homomorphism from $H(G)$ into $H(\Gamma)$ with $M(i) = i$. Then* *a) if M takes at least one of the coordinate functions into a non constant function, then there exists a function $\varphi = (\varphi_1, \varphi_2)$ from Γ into G where $\varphi_1, \varphi_2 \in H(\Gamma)$ such that*

$$M(f) = f \circ \varphi\ , \quad \textit{for all } f \in H(\Gamma)$$

i.e.,

$$M(f)(\omega) = f \circ \varphi(\omega) = f(\varphi_1(\omega), \varphi_2(\omega))\ , \quad \textit{for all } f \in H(\Gamma) \textit{ and for all } \omega \in \Gamma\ ;$$

b) further, if M is an isomorphism, $\varphi = (\varphi_1, \varphi_2)$ is a biholomorphic function from Γ onto G.

Proof. a) Let I_1^G and I_2^G denote the coordinate functions on G. Since $M(i) = i$ and M takes at least one of the coordinate functions into a nonconstant function, as in the one variable case, it is easy to show that M is linear. Let $M(I_i^G) = \varphi_i$, $i = 1, 2$. We claim that $\varphi = (\varphi_1, \varphi_2)$ maps Γ into G. To show this, let $\omega^0 \in \Gamma$ and let us consider the multiplicative linear functional m on $H(G)$ defined by

$$m(f) = M(f)(\omega^0)\ .$$

Since m is a multiplicative linear functional on $H(G)$ and G is a domain of holomorphy, by Michael's theorem [7] there exists a point $z^0 = (z_1^0, z_2^0)$ in G such that

$$m(f) = f(z^0) = M(f)(\omega^0)\ , \quad \text{for all } f \in H(G).$$

In particular, we have

$$\varphi_i(\omega^0) = M(I_i^G)(\omega^0) = m(I_i^G) = I_i^G(z^0) \quad \text{for } i = 1, 2.$$

This implies

$$\varphi \subseteq G\ .$$

Further

$$M(f)(\omega^0) = f(z^0) = f(z_1^0, z_2^0) = f(\varphi_1(\omega^0), \varphi_2(\omega^0)) = f(\varphi(\omega^0)) = (f \circ \varphi)(\omega^0)\ .$$

Thus we have

$$M(f) = f \circ \varphi, \quad \text{for all } f \in H(\Gamma)\ .$$

b) Since M is an isommorphism from $H(G)$ onto $H(\Gamma)$ the inverse map M^{-1} is also an isomorphism from $H(\Gamma)$ onto $H(G)$. Therefore, in a similar way there exist $\psi_i = M^{-1}(I_i^\Gamma)$, $i = 1, 2$, such that $\psi(G) = (\psi_1, \psi_2)(G) \subseteq \Gamma$ and $M^{-1}(f) = f \circ \psi$ for all $f \in H(\Gamma)$. But

$$I_i^G = M^{-1}(M(I_i^G)) = M^{-1}(\varphi_i) = \varphi_i \circ \psi \quad \text{for } i = 1, 2,$$

which implies

$$(I_1^G, I_2^G) = (\varphi_1 \circ \psi, \varphi_2 \circ \psi) = \varphi \circ \psi\ .$$

Thus $\varphi \circ \psi$ is the identity function on G and hence φ and ψ are biholomorphic functions.

A function $\varphi = (\varphi_1, \ldots, \varphi_n)$ is said to be conjugate biholomorphic from G onto $\bar{\Gamma}$ if $\bar{\varphi} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_n)$ is biholomorphic from G onto Γ. Now we state Ber's theorem in several variables.

Theorem. *Let G and Γ be domains of holomorphy in C^n. Then the algebras $H(G)$ and $H(\Gamma)$ are ring isomorphic if and only if there exists a function φ from G onto Γ which is either biholomorphic or conjugate biholomorphic.*

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REFERENCES

[1] Becker, J. A., Zame, W. R., *Homomorphisms into Analytic Rings*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 30 (1977), 7–10.

[2] Bers, L., *Rings of analytic functions*, Bull. Amer. Math. Soc., 54 (1948), 311–315.

[3] Burckel, R. B., Saeki, S., *Additive mappings on rings of holomorphic functions*, Proc. Amer. Math. Soc., 89 (1983), 79–85.

[4] Henriksen, M., *On the ideal structure of the ring of entire functiions*, Pacific J. Math., 2 (1952), 179–184.

[5] Krantz, S. G., *Function Theory of Several Complex Variables*, John Wiley and Sons, New York 1982.

[6] Luecking, D. H., Rubel, L. A., *Complex Analysis. A Functional Analysis Approach*, Springer-Verlag, New York 1984.

[7] Michael, E., *Locally multiplicatively-convex algebras*, AMS Memoirs, No. 11, (1952).

[8] Nandakumar, N. R., *Ring homomorphisms on* $H(G)$, Internat. J. Math. Math.Sci., 13 (1990), 393-396.

[9] Nienhuys, J. W., Thiemann, J. G. F., *On the existence of entire functions mapping countable dense subsets on each other*, Proc. Kon. Ned. Akad. van Wetensch, A 79 (1976), 331–334.

[10] Rudin, W., *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc. 78 (1955), 333–342.

STRESZCZENIE

Załóżmy, że $H(G)$, $H(\Gamma)$ są algebrami funkcji analitycznych w obszarach G, Γ płaszczyzny zespolonej.

Wykazuje się, że homomorfizm pierścieniowy algebry $H(G)$ w algebrę $H(\Gamma)$ jest bądź liniowy, bądź też antyliniowy, przy założeniu, że homomorfizm ten przeprowadza identyczność w funkcję różną od stałej.

Jako wniosek otrzymano nowy dowód twierdzenia Bersa oraz jego uogólnienie na funkcje wielu zmiennych.

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