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Department of Mathematics
Indian Institute of Technology, Kharagpur

V. K. JAIN

A Sufficient Condition for Zeros (of a Polynomial) to be in the Interior of Unit Circle

Warunek dostateczny aby zera wielomianów leżały w kole jednostkowym

Abstract. The main result of the paper is the following theorem: if $p(z)$ is a polynomial of degree n, with real coefficients, having all zeros with non-positive real part and

$$p(r) < p(R)\Big(\frac{1+r}{1+R}\Big)^{n-k}\Big(\frac{r}{R}\Big)^{k}$$

for some r, R, $0 < r < R \leq 1$, then $p(z)$ has at least $(k+1)$ zeros in $|z| < 1$.

Let $p(z) = \sum_{l=0}^{n} a_l z^l$ be a polynomial of degree n and let $M(p,r) = \max_{|z|=r} |p(z)|$. The following results concerning the size of $M(p,r)$ are well known.

Theorem A [2]. *If $p(z) = \sum_{l=0}^{n} a_l z^l$ is a polynomial of degree n, then*

$$\frac{M(p,r)}{r^n} \geq \frac{M(p,R)}{R^n}, \quad o < r < R, \tag{1.1}$$

with equality only for $p(z) = \lambda z^n$.

Theorem B [1]. *If $p(z) = \sum_{l=0}^{n} a_l z^l$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for $0 \leq r \leq R \leq 1$,*

$$\frac{M(p,r)}{(1+r)^n} \geq \frac{M(p,R)}{(1+R)^n}. \tag{1.2}$$

The result is best possible and equality holds for the polynomial $P(z) = \left(\frac{1+z}{1+R}\right)^n$.

In this note we consider certain restrictions on the estimate $M(p,r)$ and obtain the information about the zeros of the polynomial $p(z)$. More precisely, we prove

Theorem. *Let $p(z)$ be a polynomial of degree n, with real coefficients, having all zeros with non-positive real part. If, for some r, R $(0 < r < R \leq 1)$,*

$$p(r) < p(R)\Big(\frac{1+r}{1+R}\Big)^{n-k}\Big(\frac{r}{R}\Big)^{k}, \tag{1.3}$$

k, a non-negative integer, then $p(z)$ has at least $(k+1)$ zeros in $|z| < 1$. The result is best possible and the extremal polynomial is

$$p(z) = (z+1)^{n-k-1} z^{k+1} .$$

Proof of the Theorem. Suppose $p(z)$ has m zeros in $|z| < 1$ and $m \leq k$. Let $p(z) = (z-z_1)\ldots(z-z_m)(z-z_{m+1})\ldots(z-z_n)$ and assume $|z_j| < 1$ $(j = 1, 2, \ldots, m)$. Put

$$g(z) = (z-z_1)\ldots(z-z_m),$$
$$h(z) = (z-z_{m+1})\ldots(z-z_n).$$

The polynomials $p(z)$, $g(z)$ and $h(z)$ have positive coefficients. Hence, for all r, R $(0 < r < R \leq 1)$,

$$g(r) \geq g(R)\Big(\frac{r}{R}\Big)^{m} \tag{2.1}$$

by Theorem A, and

$$h(r) \geq h(R)\Big(\frac{1+r}{1+R}\Big)^{n-m} \tag{2.2}$$

by Theorem B.

On combining (2.1) and (2.2), we get

$$\begin{aligned} p(r) = g(r)h(r) &\geq g(R)h(R)\Big(\frac{1+r}{1+R}\Big)^{n-m}\cdot\Big(\frac{r}{R}\Big)^{m} \\ &= p(R)\Big(\frac{1+r}{1+R}\Big)^{n}\Big(\frac{r}{1+r}\cdot\frac{1+R}{R}\Big)^{m} \\ &\geq p(R)\Big(\frac{1+r}{1+R}\Big)^{n}\Big(\frac{r}{1+r}\cdot\frac{1+R}{R}\Big)^{k} \end{aligned}$$

a contradiction, establishing the Theorem.

For $k = n-1$ and $R = 1$, we get

Corollary 1. *If $p(z)$ is a polynomial of degree n, with real coefficients, having all zeros with non-positive real part and if for some r, $0 < r < 1$,*

$$p(r) < p(1)\Big(\frac{1+r}{2}\Big)r^{n-1}$$

then $p(z)$ has all its zeros in $|z| < 1$.

We may apply corollary 1 to the polynomial $z^n p(1/z)$ to get the following

Corollary 2. *If $p(z)$ is a polynomial with real coefficients having all zeros with non-positive real part and if for some $R > 1$*

$$p(R) < p(1)\frac{1+R}{2}$$

then $p(z)$ has no zeros in $|z| < 1$.

REFERENCES

[1] Govil, N. K., *On the maximum modulus of polynomials*, J. Math. Anal. Appl. 112 (1985), 253–258.

[2] Polya, G., Szegö, *Problems and Theorems in Analysis*, Vol. 1, p.158, Problem III 269, Berlin 1972.

STRESZCZENIE

Głównym wynikiem tej pracy jest następujące twierdzenie: jeśli $p(z)$ jest wielomianem o współczynnikach rzeczywistych, którego wszystkie zera leżą w domknięciu lewej półpłaszczyzny oraz

$$p(r) < p(R)\left(\frac{1+r}{1+R}\right)^{n-k}\left(\frac{r}{R}\right)^k$$

dla pewnych r, R, $0 < r < R \leq 1$, to $p(z)$ ma co najmniej $(k+1)$ zer w kole $|z| < 1$.

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