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Modulus Monotonic Functions

Funkcje o module monotonicznym

Abstract. Briefly, a starlike function is one for which arg $f(z)$ is increasing on $|z| = r$. Here we examine a similar concept for $|f(z)|$. Since $|f(z)|$ is periodic when $f(z)$ is single-valued, the idea must be modified. Thus, $f(z)$ is modulus monotonic on $z = re^{i\theta}$ if some interval $\alpha \le \theta \le \alpha + 2\pi$ can be decomposed into two subintervals I_1 and I_2 such that $|f(z)|$ is decreasing in I_1 and increasing in I_2.

1. Definitions. Let A be the set of all normalized functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are regular in $E : |z| < 1$. Let $\overline{A}$ be the subset of those that are also regular on $\partial E : |z| = 1$. Since Theorems about $\overline{A}$ can usually be extended to theorems about A, using a limit argument, in this work we will consider subsets of $\overline{A}$.

Definition 1. We say that $f(z)$ is modulus monotonic on the circle $|z| = r$, with angle α, if there is an α in $(-\pi/2, \pi/2)$ such that $|f(re^{i\theta})|$ is decreasing for $\theta \in I_1 : \alpha \le \theta \le \pi - \alpha$, and increasing for $\theta \in I_2 : \pi - \alpha \le \theta \le 2\pi + \alpha$.

Here we use the words increasing or decreasing to include the case that $|f(z)|$ is constant on a subset of I_1 or I_2. For fixed $r \le 1$, we let $MM(r, \alpha)$ denote subset of $\overline{A}$ of functions that are modulus monotonic on $|z| = r$ with angle α. Briefly such functions are said to be modulus monotonic. If $f(z) \in MM(r, \alpha)$ with $0 < r \le 1$, then $g(z) \equiv f(rz)/r$ is in $MM(1, \alpha)$. Hence W.L.O.G. we may concentrate our attention on the class $MM(1, \alpha)$.Notice that we have selected the arc I_1 so that I_1 is bisected by the imaginary axis. If this is not the case, we can always find a δ such that $g(z) \equiv e^{-i\delta} f(e^{i\delta} z)$ is decreasing on an arc that is bisected by the imaginary axis.

2. Elementary properties of modulus monotonic functions. Suppose that $f(z) \neq 0$ on a circle $\Gamma : z = re^{i\theta}$, $0 \le \theta \le 2\pi$. Then on Γ

$$\frac{\partial}{\partial\theta}\ln|f(z)| = \frac{\partial}{\partial\theta}\operatorname{Re}\ln f(z) = \operatorname{Re}\frac{\partial}{\partial\theta}\ln f(z) \tag{2}$$
$$= \operatorname{Re}\left[\frac{d}{dz}\ln f(z)\frac{\partial z}{\partial\theta}\right] = \operatorname{Re} i\frac{f'(z)}{f(z)}z\,.$$

Equation (2) gives

Lemma 1. *If $f(z) \in MM(r,\alpha)$ and $f(z) \neq 0$ on the circle $|z| = r$, then on that circle*

$$\operatorname{Im} z\frac{f'(z)}{f(z)} \geq 0\,, \qquad \textit{for } \alpha \leq \theta \leq \pi - \alpha \tag{3}$$

and

$$\operatorname{Im} z\frac{f'(z)}{f(z)} \leq 0\,, \qquad \textit{for } \pi - \alpha \leq \theta \leq 2\pi + \alpha\,. \tag{4}$$

Conversely, if (3) *and* (4) *hold on $|z| = r$, and $f(z) \in \overline{A}$, then $f(z) \in MM(r,\alpha)$.*

As a trivial example, consider $f(z) \equiv z$. Since $zf'(z)/f(z) = 1$, then (3) and (4) hold for every $r \neq 0$ and every α in $(-\pi/2, \pi/2)$. This example shows why we include the equal sign in (3) and (4).

Next, consider $f(z) = z + a_2z^2$. This $f(z)$ is in $MM(r,0)$ for all $a_2 > 0$. If $a_2 > 1/2$, then $f(z)$ is not univalent in E. If $a_2 > 1$, then $f(z)$ has a second zero inside E. Thus, one cannot prove that $f(z)$ is univalent, or is not zero in $0 < |z| \leq 1$, from the assumption that $f(z) \in MM(1,0)$.

3. A representation theorem. Suppose that $f(z) \in MM(1,\alpha)$. If $z = e^{i\theta}$, then $h(z) \equiv z - 2i\sin\alpha - 1/z$ gives $h(e^{i\theta}) = 2i(\sin\theta - \sin\alpha)$. If we set

$$G(z) = -h(z)z\frac{f'(z)}{f(z)} = (1 + (2i\sin\alpha)z - z^2)\frac{f'(z)}{f(z)}\,, \tag{5}$$

then on ∂E we have $\operatorname{Re} G(z) \geq 0$. But $G(z)$ has a pole with residue 1 at $z = 0$ and also poles at any other zeros of $f(z)$ in $\overline{E}$. If $f(z)$ has no zeros in $0 < |z| \leq 1$, then

$$P(z) \equiv G(z) + z - \frac{1}{z} \tag{6}$$

is regular in $\overline{E}$ and $\operatorname{Re} P(z) \geq 0$ on ∂E and hence throughout $\overline{E}$. A brief computation using (1) gives

$$P(z) = [1 + (2i\sin\alpha)z - z^2]\frac{f'(z)}{f(z)} + z - \frac{1}{z} \tag{7}$$
$$= a_2 + 2i\sin\alpha + (2a_3 - a_2^2 + 2ia_2\sin\alpha)z$$
$$+ (3a_4 + a_2^3 - 3a_2a_3 + 2i(2a_3 - a_2^2)\sin\alpha - a_2)z^2 + \dots\,.$$

We have proved

Theorem 1. *If $f(z)$ is in $MM(1,\alpha)$ and has no zeros in $0 < |z| \le 1$, then $P(z)$ defined by (7) is regular and* $\operatorname{Re} P(z) \ge 0$ *in* $\overline{E}$.

To obtain a converse to the Theorem 1, we must add some conditions on $P(z)$. From (7) we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{z(P(z) - 2i\sin\alpha)}{1 + (2i\sin\alpha)z - z^2} \tag{8}$$

Theorem 2. *Suppose that $P(0) = a_2 + 2i\sin\alpha$ and* $\operatorname{Re} P(z) \ge 0$ *in* $\overline{E}$. *If the quotient on the right side of* (8) *is regular in* $\overline{E}$, *and $f(z)$ is obtained by integrating* (7) *or* (8) *with the side conditions $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 2a_2$, then $f(z) \in MM(1,\alpha)$.*

The regularity condition on (8) implies that $P(z) - 2i\sin\alpha$ has zeros at $z_1 = e^{i\alpha}$ and $z_2 = e^{i(\pi-\alpha)}$.

Corollary 1. *If $f(z)$ given by* (1) *is in $MM(1,\alpha)$ and has no zeros in $0 < |z| \le 1$, then* $\operatorname{Re} a_2 \ge 0$. *If Re $a_2 = 0$, then $f(z) \equiv z$.*

Theorems 1 and 2 suggest several open questions. If $f(z) \in MM(1,\alpha)$ and $0 < r < 1$, is $f(z) \in MM(r,\beta)$ for some suitable β? It is clear that in general $\beta \ne \alpha$, and indeed β depends on r.

What functions $\beta(r)$ are admissable when $f(z) \in MM(1,\alpha)$? What conditions on $P(z)$ are necessary and sufficient for $f(z)$ to be univalent in E?

We next consider the special case $\alpha = 0$. Then equation (7) becomes

$$\begin{aligned} P(z) &= (1-z^2)\frac{f'(z)}{f(z)} - \frac{1-z^2}{z} \\ &= a_2 + (2a_3 - a_2^2)z + (3a_4 + a_2^3 - 3a_2a_3 - a_2)z^2 + \dots \end{aligned} \tag{9}$$

and equation (8) becomes

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{1-z^2}P(z) \tag{10}$$

Suppose that in (10) we put $P(z) = 2 + iz - iz^2$ and replace $f(z)$ by $f_1(z)$. Then for $z = e^{i\theta}$

$$\operatorname{Im} z\frac{f_1'(z)}{f_1(z)} = \frac{2 - \sin\theta + \sin 2\theta}{2\sin\theta} \tag{11}$$

Therefore, $zf_1'(z)/f_1(z)$ satisfies the conditions (3) and (4) of Lemma 1 on ∂E when $\alpha = 0$. With this $P(z)$ equation (11) gives

$$f_1(z) = z\frac{1+z}{1-z}\exp(iz - i\ln(1+z))\,. \tag{12}$$

Hence $|f_1(e^{i\theta})|$ is decreasing for $0 < \theta < \pi$ and increasing for $\pi < \theta < 2\pi$. It is important to observe that in this example the coefficients are not all real and that $f_1(z) \notin MM(r,0)$ for any r in $(0,1)$. Although $f_1(z)$ is not regular at $z = \pm 1$, this example indicates that $f(z) \in MM(1,0)$ does not imply that the a_n are all real or that $f(z) \in MM(r,0)$ for any $r \in (0,1)$. To obtain such conclusions we must consider an appropriate subset.

Definition 2. A function $f(z)$ in A with all coefficients real is said to be modulus monotonic on $|z| = r$ with real coefficients if $f(z) \in MM(r,0)$. We let $MMR(r)$ denote the set of all such functions for fixed r in $(0,1)$.

Briefly, $f(z) \in MMR(r)$ if all the coefficients are real and the inequalities (3) and (4) are satisfied on $|z| = r$ with $\alpha = 0$.

Theorem 3. *Suppose that $f(z) \in MMR(1)$ and $f(z) \neq 0$ for $0 < |z| \leq 1$. Then $f(z) \in MMR(r)$ for each r in $(0,1)$.*

Proof. Since $f(z)$ has real coefficients, $\operatorname{Im} zf'(z)/f(z) = 0$ if $-1 \leq z \leq 1$. But $\operatorname{Im} zf'(z)/f(z)$ is a harmonic function that is nonnegative on $z=e^{i\theta}$, $0 \leq \theta \leq \pi$. Hence, it is nonnegative throughout the upper half of the unit disk. A similar argument shows that $\operatorname{Im} zf'(z)/f(z) \leq 0$ throughout the lower half of the unit disk.

We return to equations (9) and (10). If $f(z)$ has all coefficients real, the same is true of $P(z)$. If $a_2 = 0$, then $P(z) \equiv 0$ and $f(z) \equiv z$. If $a_2 \neq 0$, then $p(z) \equiv P(z)/a_2$ is normalized by $p(0) = 1$, has all coefficients real and has positive real part in E. Consequently, from well-known properties of typically-real functions [2, 1 vol. I p.185], the function

$$T(z) = \frac{1}{a_2}\left[z\frac{f'(z)}{f(z)} - 1\right] = \frac{z}{1-z^2}\,\frac{P(z)}{a_2} = \frac{z}{1-z^2}\,p(z) \tag{13}$$

is typically-real, with $T(z) = z + \ldots$. This gives

Theorem 4. *If $f(z) \in MMR(1)$ and $f(z) \neq 0$ for $0 < z \leq 1$, then $T(z)$ defined by (13) is typically-real in E. Conversely, if $T(z)$ is typically-real, and $f(z)$ is the solution of (13) with $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 2a_2$, then $f(z) \in MMR(r)$ for each r in $(0,1)$, and $f(z) \neq 0$ for $0 < |z| > 1$.*

4. Coefficient bounds. Whenever a new class of analytic functions is introduced, it is customary to look for sharp bounds for the coefficients $|a_n|$. Perhaps the most famous result of this type is DeBranges' Theorem which gives $|a_n| \leq n$ for all n if $f(z) \in S$.

Consequently, it is something of a surprise that in the class $MM(1,0)$ the coefficient $|a_n|$ has no upper bound for any $n > 1$. However, if we fix $a_2 > 0$, then $|a_n|$ can be bounded. This is the content of

Theorem 5. *If $f(z) \in MM(1,0)$ and $f(z) \neq 0$ for $0 < |z| \leq 1$, and $a_2 > 0$, then for each $n > 2$ we have $|a_n| \leq A_n$ where A_n is defined by*

$$F(z) \equiv z \exp \frac{a_2 z}{1-z} = z + \sum_{n=2}^{\infty} A_n z^n . \tag{14}$$

Further, this upper bound is best possible for each $n > 2$.

Proof. We use the technique of dominant power series [1, vol. I, pp. 82–83] and the associated symbol $\ll$. From Corollary 1 we may assume that $a_2 > 0$. Then from Carathéodory's Theorem for functions with positive real part in E [1, vol. I, p. 77–81] equations (9) and (10) give

$$(15)\qquad \frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{1-z^2}\,P(z) \ll \frac{a_2}{1-z^2}\cdot\frac{1+z}{1-z}\,.$$

If we integrate (15) from 0 to z we obtain

$$(16)\qquad \ln\frac{f(z)}{z} \ll a_2\frac{z}{1-z}$$

and hence $f(z) \ll z\exp a_2 z/(1-z)$. For example, expanding the right side of (14) gives

$$|a_3| \le a_2 + \frac{1}{2}\,a_2^2\,, \quad |a_4| \le a_2 + a_2^2 + \frac{1}{2}\,a_2^3\,.$$

Now $F(z)$ is not in $MM(1,0)$ because it is unbounded at $z = 1$. But for $r < 1$, the function $z\exp a_2 z/(1-rz)$ is in $MM(1,0)$ and the nth coefficient for this function can be made arbitrarily close to A_n by selecting r close to 1.

If $\alpha \ne 0$, we can also obtain coefficient bounds, but in this case the bounds are far from best possible. For brevity set $\eta = e^{-i\alpha}$. Then equation (7) can be put in the form

$$P(z) - 2i\sin\alpha = (1+\eta z)(1-\bar\eta z)\frac{f'(z)}{f(z)} - \frac{(1+\eta z)(1-\bar\eta z)}{z} = q(z) = a_2 + \sum_{n=1}^{\infty} q_n z^n\,,$$

where $\operatorname{Re} q(z) \ge 0$ in $\overline{E}$. Consequently,

$$(17)\qquad \frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{q(z)}{(1+\eta z)(1-\bar\eta z)} = \frac{1}{2\cos\alpha}\left[\frac{\eta}{1+\eta z} + \frac{\bar\eta}{1-\bar\eta z}\right]q(z)$$

and (losing all hope of a sharp result)

$$(18)\qquad \frac{f'(z)}{f(z)} - \frac{1}{z} \ll \frac{1}{2\cos\alpha}\,\frac{2}{1-z}\,a_2\frac{1+z}{1-z}\,.$$

Integrating from 0 to z we obtain

$$(19)\qquad \ln\frac{f(z)}{z} \ll \frac{a_2}{\cos\alpha}\int_0^z \frac{1+t}{1-t^2}\,dt = \frac{a_2}{\cos\alpha}\left[\frac{2}{1-z} + \ln(1-z) - 2\right]\,.$$

We have proved

Theorem 6. *If $f(z) \in MM(1,\alpha)$ and $f(z) \ne 0$ for $0 < |z| \le 1$, and $a_2 > 0$, then $|a_n| \le A_n$, for each $n > 2$, where A_n is defined by*

$$(20)\qquad F(z) = z\exp\left[\frac{a_2}{\cos\alpha}\left[\frac{2z}{1-z} + \ln(1-z)\right]\right] = z + \frac{a_2}{\cos\alpha}\,z^2 + \sum_{n=3}^{\infty} A_n z^n\,.$$

We observe that in the definition of the class $MM(1,\alpha)$, we could change the arcs I_1 and I_2 by asking that $|f(e^{i\theta})|$ is decreasing for $0 \leq \theta \leq \beta$ and increasing for $\beta \leq \theta \leq 2\pi$. However, the equations we obtain with this selection of arcs are much less pleasant.

REFERENCES

[1] Goodman, A. W., *Univalent Functions*, Polygonal Publishing House, Washington, New Jersey.

[2] Rogosinski, W., *Über positive harmonische Entwicklungen und typischreele Potenzreihen*, Math. Z. 35 (1932), 93–121.

After my paper was submitted, I learned that Prof. Yusuf Avci had been working on the same topic, but only Theorem 1 is in the intersection of our results. The appropriate references to his work are:

1. *Univalent functions with the monotonic modulus property*, Complex Variables, Theory and Applications, 10 (1988), 161–169.
2. *Further results on the univalent functions with the monotonic modulus property*, Ann. Polon. Math., 53 (1991), 57–60.
3. *Univalent meromorphic functions with the monotonic modulus property*, Proc. 2nd Nat. Math. Sym. Turkish Math. Soc. (1989), 351–356.
4. *On monotone meromorphic functions*, Proc. 3rd Nat. Math. Sym. Turkish Math. Soc. (1990), to appear.

STRESZCZENIE

Jeśli f jest funkcją gwiaździstą w kole jednostkowym E, to arg $f(z)$ jest funkcją rosnącą dla $|z| = r < 1$. W pracy tej badamy analogiczne zagadnienie dla $|f(z)|$. Ponieważ $|f(z)|$ jest funkcją okresową dla funkcji f jednoznacznej, więc zagadnienie należy zmodyfikować. Zatem $f(z)$ ma monotoniczny moduł dla $z = re^{i\theta}$ jeśli pewien przedział $\alpha \leq \theta \leq \alpha + 2\pi$ da się rozłożyć na dwa podprzedziały I_1, I_2 tak, że $|f(z)|$ maleje w I_1 oraz rośnie w I_2.

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