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On Integral Means of the Convolution

Średnie całkowe dla splotów

Abstract. Let $f * g$ denote the convolution of two functions holomorphic in the unit polydisc U^n. We prove the following theorem: If $1 \le p \le s \le q$ and $f \in H^p$, $g \in H^q$ then

$$\|f * g\|_s \le \|f\|_p \cdot \|g\|_s .$$

Besides, if $e(z) = \sum_\alpha z^\alpha$ then $\widetilde{H}^p = \{\widetilde{f}(z) + te(z)\ ,\ f \in H^p,\ t \in \mathbf{C}\}$ is a commutative Banach algebra with the unit element e and H^p is its maximal ideal.

Let U be the open unit disc in the complex plane $\mathbf{C}$ and let T be its boundary. The unit polydisc U^n and the torus T^n are the product of n copies of U and T, respectively. We assume throughout that μ is a positive (σ-finite) measure, normalized so that $\mu(T^n) = 1$.

For $0 < p < \infty$ let H^p be the class of all complex-valued functions f holomorphic in U^n for which

$$\|f\|_p = \sup_{0<r<1} M_p(r, f) < \infty ,$$

where

$$M_p(r, f) = \left(\int_{T^n} |f(rz)|^p \, d\mu(z) \right)^{1/p} .$$

Since $|f|^p$ is n-subharmonic, the supremum can be replaced by the limit as $r \to 1^-$; H^∞ is the space of all functions f bounded and holomorphic in U^n; $\|f\|_\infty = \sup_{z \in U^n} |f(z)|$.

The convolution (or Hadamard product) of two functions f, g holomorphic in U^n is the function $f * g$ defined by the following formula

$$(f * g)(r^2 z) = \int_{T^n} f(r\zeta) g(rz\overline{\zeta}) \, d\nu(\zeta)\ ,\quad 0 < r < 1\ ,\quad z \in U^n$$

where $z \cdot \zeta = (z_1\zeta_1, \ldots, z_n\zeta_n)$.

If $f(z) = \sum_\alpha a_\alpha z^\alpha$, $g(z) = \sum_\alpha b_\alpha z^\alpha$, where α ranges over multi-indices, are holomorphic in U^n, then

$$(f * g)(z) = \sum_\alpha a_\alpha b_\alpha z^\alpha \ , \quad z \in U^n \ .$$

In his paper [1] Boo Rim Choe gave an integral mean inequality for the convolution of functions in the case $p \in (0,1)$; (see [2], too).

In this note we prove the following

Theorem 1. *If $1 \le p \le s \le q$, and $f \in H^p$, $g \in H^q$ then*

$$\|f * g\|_s \le \|f\|_p \cdot \|g\|_s \ . \tag{1}$$

Let us observe that the inequality (1), in some sense, corresponds to the Young generalized inequality, [4].

Proof. Let λ be a fixed number, $\lambda \ge 1$. Then

$$M^p_{\lambda p}(r^2, f * g) = \left[\int_{T^n} |(f * g)(r^2 z)|^{p\lambda} \, d\mu(z) \right]^{1/\lambda} = $$

$$= \left[\int_{T^n} \left| \int_{T^n} f(r\zeta) g(rz \cdot \bar{\zeta}) \, d\nu(\zeta) \right|^{p\lambda} d\mu(z) \right]^{1/\lambda} .$$

Using the Minkowski integral inequality [4] we obtain

$$M^p_{\lambda p}(r^2, f * g) \le \left[\int_{T^n} \left(\int_{T^n} |f(r\zeta) g(rz \cdot \bar{\zeta})|^{p\lambda} d\mu(z) \right)^{\frac{1}{p\lambda}} d\nu(\zeta) \right]^p =$$

$$= \left[\int_{T^n} |f(r\zeta)| \, d\nu(\zeta) \left(\int_{T^n} |g(rz \cdot \bar{\zeta})|^{p\lambda} d\mu(z) \right)^{\frac{1}{p\lambda}} \right]^p \le$$

$$\le \int_{T^n} |f(r\zeta)|^p \, d\nu(\zeta) \cdot \left[\int_{T^n} |g(rz \cdot \bar{\zeta})|^{p\lambda} d\mu(z) \right]^{\frac{1}{\lambda}} \le$$

$$\le \|f\|^p_p \cdot \|g\|^p_{p\lambda}$$

for $1 \le \lambda p \le q$. Since $M_\lambda(r^2, |h|^p) = M^p_{\lambda p}(r^2, h)$ our Theorem is proved.

Now, let us remark, that a Banach algebra is a linear algebra with a Banach space norm which is related to the multiplication by $\|xy\| \le \|x\| \, \|y\|$.

The space H^p, $p \ge 1$, is a Banach space [3]. Thus, from Theorem 1 we see that H^p, $p \ge 1$, is a Banach algebra. Let us notice that H^p does not contain a unit element.

Suppose $e(z) = \sum_\alpha z^\alpha$. We see that $e \notin H^p$. Let us consider

$$\tilde{H}^p = \{ \tilde{f}(z) = f(z) + t \cdot e(z) : f \in H^p \ , \ t \in \mathbf{C} \} \ ;$$

$$\|\tilde{f}\|_p = \|f\|_p + |t| \ .$$

Then for $\widetilde{f}(z) = f(z) + te(z) \in \widetilde{H}^p$ and $\widetilde{g}(z) = g(z) + se(z) \in \widetilde{H}^p$ we have

$$(\widetilde{f} * \widetilde{g})(z) = (f * g)(z) + sf(z) + tg(z) + tse(z) .$$

Moreover,

$$\|\widetilde{f} * \widetilde{g}\|_p \le \|f * g\|_p + |s| \cdot \|f\|_p + |t| \cdot \|g\|_p + |ts| \le \|\widetilde{f}\|_p \cdot \|\widetilde{g}\|_p .$$

Thus we have

Proposition. *$\widetilde{H}^p$, $p \ge 1$ is a commutative Banach algebra with the unit element e.*

Theorem 2. *H^p is a maximal ideal of $\widetilde{H}^p$.*

Proof: It is well-known, that for A being a commutative algebra with the unit element J is a maximal ideal iff A/J is a field. Let us notice that $\widetilde{H}^p/H^p$ is the field $\mathbf{C}$.

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STRESZCZENIE

Niech $f * g$ oznacza splot dwóch funckji holomorficznych w polidysku U^n. Dowodzimy następującego twierdzenia: jeśli $1 \le p \le s \le q$, oraz $f \in H^p$, $g \in H^q$ to

$$\|f * g\|_s \le \|f\|_p \cdot \|g\|_s .$$

Ponadto, jeśli $e(z) = \sum_\alpha z^\alpha$ to $\widetilde{H}^p = \{\widetilde{f}(z) + te(z)\ ,\ f \in H^p,\ t \in \mathbf{C}\}$ jest przemienną algebrą Banacha z elementem jednostkowym e i H^p jest jej maksymalnym ideałem.

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