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Strong Limit Theorems for the Growth of Increments of Additive Processes in Groups. Part I. Basic Properties of Globular Sets.

Mocne prawa graniczne dla przyrostów procesów addytywnych w grupach Część I. Podstawowe własności zbiorów globularnych

Abstract. In the paper the notion of a globular set is introduced and a wide class of groups with globular neighbourhoods of zero is described. Next, various criteria for families of globular neighbourhoods of zero to be upper and lower classes of sets for increments of group-valued additive processes are given.

We first explain more precisely the main ideas of our approach.

Let G be an Abelian topological group. An open Baire neighbourhood U of zero in G is called globular, if there exists a sequence of open Baire sets $U = U(0) \supseteq U(-1) \supseteq U(-2) \supseteq \ldots$ containing zero such that for each $k \geq 1$ there are at most countable families of Baire sets $\{C_j(k)\}$ and $\{H_j(k)\}$ satisfying the conditions: for a fixed $k \geq 1$ the sets $C_j(k)$ are disjoint, $U^c(-k) = \bigcup_j C_j(k)$, $C_j(k) + H_j(k) \subseteq U^c(-k-1)$ and $H_j^c(k) \subseteq -H_j(k)$. An arbitrary globular set may be obtained by a translation of $U \ni 0$. It can be shown that each open convex set in a locally convex linear topological Lindelöf space is globular. Suppose $\{X(t)\ ,\ t \in R_+^q\}$ is a symmetric additive stochastic process taking values in G and $W = < w, z > \subset R_+^q$ is a bounded rectangle, where R_+ denotes the set of nonnegative real numbers and $q \geq 1$. Then the following maximal symmetrization inequality can be proved:
for every globular set $U \subset$ G,

$$P\Big[\bigcup_{s,t \in Q(W)} \{\Delta X(< s, t)) \notin U\}\Big] \leq 4^q P\Big[\Delta X(W) \notin U(-2q)\Big]\,,$$

where $Q(W)$ is a countable dense subset of W. Based on this estimate several results containing integral tests for increasing families of globular sets $\{U_t\}$ are obtained, ensuring that $\{U_t\}$ are upper or lower classes of sets for the growth of increments $\Delta X(< 0, t))$ on large and small rectangles. As corollaries, limit theorems for additive processes with values in linear topological spaces are given. It appears that torus and some cyclic groups do not contain any globular sets. Therefore representations of additive processes in torus and cyclic groups taking values in suitable globular groups are also described, and by means of these representations limit theorems for such processes are derived.

1. Introduction. Investigation of basic properties of stochastic processes with independent increments has attracted attention of probabilists for the past sixty years. The systematic study of the class of processes with independent increments was originated by de Finetti [5], and a remarkable progress in this area was made next by Kolmogorov and Lévy. Kolmogorov [19] described the characteristic function of a one-parameter process with independent increments having finite variance, and Lévy [22] obtained the characteristic function of such a process in the general case. Lévy [22] also proved that every process with independent increments is the sum of some nonrandom function and two other independent processes: purely discrete and stochastically continuous. Moreover, he showed that a stochastically continuous process with independent increments has a modification without discontinuities of the second kind. These investigations were continued by Itô [15], who obtained a decomposition of a stochastically continuous process with independent increments into two independent parts: Gaussian and Poissonian, and established a representation of the last process by means of a stochastic integral. Local and asymptotic growth at infinity of trajectories for processes with independent increments was investigated by Khintchine [18] and Gnedenko [8,9].

Further advances in the research work devoted to multi-parameter stochastic processes with independent increments (called in short additive) were made by Katkauskaité [17] and Adler, Monrad, Scissors and Wilson [1]. The last paper, among other things, gives the characteristic functions and the Lévy-Itô path decomposition for real-valued additive processes. Investigations concerning certain local and asymptotic properties of trajectories for multi-parameter real-valued additive processes can be found in a paper by Zinčenko [31].

The comprehensive list of references concerning this subject cannot be presented here. Some more information is furnished in an expository paper by Adler et al. [1] or the books devoted to the general theory of stochastic processes by Gikhman and Skorohod [6,7]. A survey of the most important properties of one-parameter group-valued processes with independent increments and further references are given in the monographs by Heyer [13] and Skorohod [25].

The aim of this article is the study of some fundamental limiting properties which characterize realizations of group-valued additive processes indexed by the multi-dimensional set of parameters $R_+^q, R_+ =< 0, \infty)$, $q \geq 1$. Namely, we investigate the speed of the growth of q-dimensional increments for trajectories of additive processes on asymptotically infinite and negligibly small rectangles $V \subset R_+^q$. This topic of investigations has been raised in view of the following situation. The law of the iterated logarithm for Banach space valued Brownian motion and related theorems for i.i.d. random elements via invariance principles were studied extensively during the past decade (see e.g. Goodman and Kuelbs [11] and references therein). On the other hand, there are fairly well-known strong limit theorems for real-valued processes with independent increments which imply the law of the iterated logarithm for real Brownian motion, also in its functional form (c.f. Gikhman and Skorohod [6], Chapter VI, §6, and Wichura [29]). However, this fact has no counterpart in the Banach space case, or more generally – in topological vector spaces and groups. Thus the present work is perhaps the first attempt to fill this gap.

The main tool in our approach is a modified version of symmetrization inequality. Presumably the most general form of such a result is known for processes or random elements falling outside of convex sets in a Banach space (Gikhman and Skorohod [7], Chapter IV, §1) or balls in a normed linear space (Vakhania, Tarieladze and Chobanian [28], Chapter V, §2). However, neither of these kinds of sets can be defined at all in an arbitrary topological group, and therefore in the case of group–valued random elements no results of this type are known.

To remove these difficulties we introduce in Section 2 the notion of a globular set in a T_0 topological Abelian group and quote a result on the existence of sufficiently rich families consisting of globular sets for a wide class of spaces. The definition of a globular set enables us to establish the so-called maximal symmetrization inequality given in Lemma 3.2 and its Corollaries of Section 3. The mentioned estimate provides an upper bound for probability of escape from a globular set of the process over a bounded rectangle expressed in terms of distribution of the increment of the process on the considered rectangle. On the basis of this result we obtain in Section 4 limit theorems for local and global growth of increments of additive processes on infinitely small and very large subintervals of the parameter set. Recent developments in the theory of strong convergence for some classes of processes with independent increments suggest that local properties and asymptotic behaviour at infinity of their trajectories need not be investigated separately, because they are really of the same nature (see e.g. Mueller [24]). Therefore we prove local and global limit theorems simultaneously. The asymptotic behaviour of additive processes is described by means of limits of the form

$$\liminf_{t\to 0 \text{ or } \infty} D_t = \bigcup_{0<T_1<T_2} \bigcap_{t\notin <T_1,T_2>} D_t\,,$$

and

$$\limsup_{t\to 0 \text{ or } \infty} D'_t = \bigcap_{0<T_1<T_2} \bigcup_{t\notin <T_1,T_2>} D'_t$$

for some random events D_t and D'_t, $t \in R^q_+$. In a typical situation we have $D_t = \{\Delta X(<0,t)) \in U_t\}$ and $D'_t = \{\Delta X(<0,t)) \notin U'_t\}$, where $\{U_t\}$ and $\{U'_t\}$ are increasing families of globular sets and $\Delta X(V)$ denotes the increment of the process X on a rectangle $V \subset R^q_+$. Then the assertion

$$(*) \qquad P\Big[\liminf_{t\to 0 \text{ or } \infty} D_t\Big] = 1$$

may be interpreted as follows: outside of a (random) interval $< T_1, T_2 > \subset R^q_+ \setminus \partial R^q_+$ the increments $\Delta X(<0,t))$ of the process in question stay eventually with probability 1 in sets U_t, $t \in R^q_+ \setminus < T_1, T_2 >$. Similarly,

$$(**) \qquad P\Big[\limsup_{t\to 0 \text{ or } \infty} D'_t\Big] = 1$$

means that with probability 1 for every bounded interval $< T_1, T_2 > \subset R^q_+ \setminus \partial R^q_+$ there exist points $t \in R^q_+$, $t \notin < T_1, T_2 >$ for which $\Delta X(<0,t)) \notin U'_t$. In this manner various types of strong laws of large numbers (such as Marcinkiewicz–Zygmund SLLN) as well as the law of the iterated logarithm may be treated. To see this it suffices to consider the one-dimensional case and take

$$U_t = t^{1/r} \cdot (-\varepsilon, \varepsilon) , \quad 0 < r < 2 , \text{ or}$$
$$U_t^{(\prime)} = (2t \log\log|t|)^{1/2} \cdot (-1 - (+)\varepsilon, 1 + (-)\varepsilon) , \quad \varepsilon > 0 .$$

Also the rate of convergence can be described in this way. As an example of applications we discuss in greater detail the law of the iterated logarithm for Brownian surfaces with values in locally convex linear topological spaces. In fact we consider a more general situation accepting instead of R_+^q a partially bounded away from zero or infinity set B to obtain integral tests for relations like (*) and (**) when t varies in B. Next considering specific cases of B we can obtain local, global and the so-called "mixed" law of the iterated logarithm.

Unfortunately, some important classes of groups have none globular neighbourhoods of zero. Therefore Section 5 is devoted to a brief discussion of a particular case of a group of such kind, namely torus. In this section we obtain a representation in an appropriate globular group for an additive process taking values in torus. Next, based on the mentioned representation we prove various local limit theorems for additive processes in torus.

In view of regulations concerning the size of articles in this journal the paper is divided into parts, and the first part contains three sections. The second part of the article will appear in the next issue of Annales.

2. Globular sets. In this section we explain the concept of globular sets and present basic properties of these objects.

Let G be an Abelian topological group.

Definition 2.1. An open Baire set $U \ni 0$ in G is called globular, if there exists an open Baire neighbourhood of zero $U(-1) \subseteq U = U(0)$, such that

$$(2.1)\quad \begin{cases} U(0)^c = \bigcup_j C_j \,, \text{ where } \{C_j = C_j(0) \,,\ j \in J\} \text{ is at most countable family} \\ \text{of disjoint Baire sets;} \end{cases}$$

$$(2.2)\quad \begin{cases} \text{to each } C_j \text{ there corresponds a Baire set } H_j = H_j(0) \text{ satisfying} \\ \text{the conditions: } C_j + H_j \subseteq U(-1)^c \text{ and } H_j^c \subseteq -H_j \,. \end{cases}$$

Moreover, $U(-1)$ possesses the same properties as U specified by (2.1) and (2.2) with possibly another sets C_j and H_j.

In other words, an open Baire neighbourhood U of zero is globular if there exists a sequence of open Baire neighbourhoods of zero $U = U(0) \supseteq U(-1) \supseteq \ldots$ such that for every $k \geq 1$, $U(-k)$ satisfy conditions (2.1)–(2.2) with $C_j = C_j(k)$, $H_j = H_j(k)$ and $U(-1)$ replaced by $U(-k-1)$. An arbitrary open Baire set $U' \ni x$ is called globular if $U' = x + U$, where U is a globular neighbourhood of zero. In such a case we put $U'(-k) = x + U(-k)$, $C_j'(k) = x + C_j(k)$ and $H_j'(k) = H_j(k)$.

The notion of a globular set seems to be quite new, therefore we have to discuss basic properties of this concept. The below properties show that under suitable additional conditions globular sets can be taken as a basis at zero for the topology of $\mathbf{G}$.

1°. *The product of two globular sets is globular.*

Proof. Let $U'(-1)$, $\{C'_j\}$, $\{H'_j\}$ and $U''(-1)$, $\{C''_i\}$, $\{H''_i\}$ for $j \in J$ and $i \in I$ be the families of Baire sets associated with two globular sets U' and U'' according to Definition 2.1. For $U = U' \cap U''$ we put $C_j = C'_j$, $j \in J$, $C_i = C''_i \setminus \bigcup_j C'_j$, $i \in I$ (if some sets C_i are empty, the corresponding indices are rejected), $H_j = H'_j$, $j \in J$, $H_i = H''_i$, $i \in I$, and $U(-1) = U'(-1) \cap U''(-1)$. Then conditions (2.1)–(2.2) are obviously fulfilled, and $U'(-1) \cap U''(-1)$ can be treated likewise $U' \cap U''$.

2°. *If U is a globular set, then $-U$ is also globular.*

Proof. Let U be a globular set, and let $\{C_j\}$, $\{H_j\}$ and $U(-1)$ be Baire sets as specified in Definition 2.1 for U. Then $\{-C_j\}$, $\{-H_j\}$ and $-U(-1)$ are also Baire sets, and conditions (2.1)–(2.2) for these sets with U replaced by $-U$ can be easily verified.

From 1° and 2° we conclude at once that for every globular neighbourhood of zero U there exists a symmetric globular neighbourhood of zero V contained in U, namely $V = U \cap (-U)$.

3°. *Every open Baire set V containing any globular set U is globular.*

Proof. Let $\{C_j\}$, $\{H_j\}$ and $U(-1)$ be the Baire sets corresponding to the globular set U. Put $C'_j = C_j \cap V^c$, $H'_j = H_j$ and $V(-1) = U(-1)$. Then these sets satisfy (2.1)–(2.2) with V instead of U.

4°. *Let $\{U_i\}$ be a countable family of globular sets. Then their union $U = \bigcup_i U_i$ is a globular set too.*

Proof. This is an immediate consequence of 3°.

5°. *Let U be a globular set and let A be an arbitrary set such that $A + U$ is a Baire set. Then $A + U$ is a globular set. In particular, this is the case when A is at most countable.*

Proof. This follows easily from 3° and 4°.

6°. *A family $\mathcal{U} = \{U\}$ of globular neighbourhoods of zero in an Abelian topological group $\mathbf{G}$ constitutes a basis at zero for a topology making $\mathbf{G}$ a topological group, if and only if*

(i) *for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$, and*

(ii) *for each $U \in \mathcal{U}$ and $x \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $x + V \subseteq U$.*

Proof. An arbitrary basis at zero for every topological group clearly satisfies (i)–(ii), so it is enough to demonstrate the converse assertion. By virtue of 1° and 2°, $\mathcal{U}$ possesses the finite intersection property. Furthermore, if $U \in \mathcal{U}$, then $V = U \cap (-U)$

is globular and $V = -V \subseteq U$. Taking into account Theorem 4.5 of Chapter II in Hewitt and Ross [12] we see that conditions (i)-(ii) ensure that $\mathcal{U}$ is an open basis at zero making $\mathbf{G}$ a topological group.

A T_0 topological Abelian group $\mathbf{G}$ having as a local basis a family $\mathcal{U}$ consisting of globular neighbourhoods of zero may be termed as a globular topological group. The next result shows that the class of such groups is sufficiently wide.

Proposition 2.2. *Let $\mathbf{G}$ be a linear topological space (real or complex) and let U be a convex open neighbourhood of zero in $\mathbf{G}$. If U^c equipped with a topology containing open Baire sets is a Lindelöf space, then U is globular.*

The proof of this result is based on the general Hahn–Banach theorem and will be given elsewhere. In fact we are interested in a situation when the class of globular subsets of $\mathbf{G}$ is sufficiently rich, but globular neighbourhoods of zero need not form necessarily a local basis for the topology of $\mathbf{G}$. As we shall see in Section 4, in such a case we are able to give various integral tests for families of globular sets $\{U_t\}$ to be upper or lower classes of sets for increments $\Delta X(<0,t))$ of an additive process in $\mathbf{G}$, $<0,t) \subseteq R^q_+$, though they do not describe convergence of $\Delta X(<0,t))$ to zero.

Examples.

1. Each open convex set U containing zero in a linear topological Lindelöf space is globular, because a closed subspace of the Lindelöf space is Lindelöf. In particular, every open ball in a separable Banach or Hilbert space is globular. Consequently, every locally convex linear topological Lindelöf space is a globular topological group.

2. In certain cases we can take $U = U(-1) = U(-2) = \dots$. To see this, let $S \neq \emptyset$ be an arbitrary parameter set and let $\mathbf{R}^S$ be the product space equipped with the Tychonoff topology. Then every open neighbourhood of zero in $\mathbf{R}^S$ of the form

$$U = \{x \in \mathbf{R}^S : -\varepsilon_1' < x_{s_1} < \varepsilon_1, \dots - \varepsilon_n' < x_{s_n} < \varepsilon_n\} ,$$

where $0 < \varepsilon_1, \varepsilon_1', \dots \varepsilon_n, \varepsilon_n' \leq \infty$, $n \geq 1$, is globular. This can be shown in a direct way. Indeed,

$$U^c = \bigcup_{j \leq n} \{x \in \mathbf{R}^S : x_{s_j} \geq \varepsilon_j\} \cup \bigcup_{j \leq n} \{x \in \mathbf{R}^S : x_{s_j} \leq -\varepsilon_j'\} .$$

Define $C_1 = \{x_{s_1} \geq \varepsilon_1\}$, $C_j = \bigcap_{k<j} C_k^c \cap \{x_{s_j} \geq \varepsilon_j\}$ for $j \leq n$, $C_m = \bigcap_{k<m} C_k^c \cap \{x_{s_{m-n}} \leq -\varepsilon_{m-n}'\}$ for $n < m \leq 2n$, $H_j = \{x_{s_j} \geq 0\}$ for $j \leq n$, and $H_m = \{x_{s_{m-n}} \leq 0\}$ for $n < m \leq 2n$. As can be easily seen, $\{C_j\}$ and $\{H_j\}$ satisfy (2.1) and (2.2) with $U(-1) = U$.

3. Let $\mathbf{Z}$ be the set of integers with the usual addition of real numbers. Clearly, $\mathbf{Z}$ is a globular subgroup of the group $\mathbf{R}$. More generally, let $\mathbf{Z}(r)$ be the set of numbers $\{0, \pm 1/r, \pm 2/r, \dots\}$, where $r \in \mathbf{N} = \{1, 2, \dots\}$ and let $\mathbf{Z}(r_1, \dots r_p) = \mathbf{Z}(r_1) \times \dots \times \mathbf{Z}(r_p)$ be the product group considered with addition of vectors. Then $\mathbf{Z}(r_1, \dots r_p)$ is a globular subgroup of the globular group $\mathbf{R}^p$.

4. Notice that for a class of groups globular neighbourhoods of zero do not exist at all. For instance, this is the case when $\mathbf{G}$ is equal to the torus $\mathbf{T}_p = \{z \in C : |z| = 1\}^p$, $p \in \mathbf{N}$ regarded with coordinate-wise multiplication of complex numbers, or $\mathbf{G} = \mathbf{C}(p) = \{z \in C : z^p = 1\}$ with p even. However, in a cyclic group $\mathbf{C}(p)$ with p odd, globular neighbourhoods of zero constitute a local basis for the topology of $\mathbf{C}(p)$, in particular the one-point set $\{0\}$ is globular. Consequently, the group $\mathbf{C}(r_1, \ldots r_p) = \mathbf{C}(r_1) \times \ldots \times \mathbf{C}(r_p)$ is globular if and only if all r_i are odd, and it is not globular iff at least one of these numbers is even.

3. Maximal symmetrization inequality. In this section we present an inequality which allows us to estimate the behaviour of realizations for a group-valued stochastic process over a bounded rectangle $< w, z > \subset R_+^q$ by its properties on the boundary $\partial < w, z >$.

Let T be a subset of R_+^q and let $X_T = \{X(t)\ ,\ t \in T\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ taking values in a T_0 topological Abelian group $\mathbf{G}$ equipped with its Baire σ-field $\mathcal{G}(\mathbf{G}) = \mathcal{G}$. From now on, throughout the whole paper we impose a general restriction on the class of considered random elements assuming that group operations within it are always permitted. In particular, this is substantiated if the process X satisfies one of the conditions 1°, 2° or 3° of an earlier work by Zapała [30]. In the case of 3° we assume in addition that distributions of finite vectors $\overrightarrow{X}_L = \big(X(t(1)), \ldots X(t(L))\big)$, $t(1), \ldots t(L) \in T$, are determined uniquely on $\mathcal{G}(\mathbf{G}^L)$ by their restrictions to $\mathcal{G}^L$.

In the sequel the index set $T \subset R_+^q$ is a set satisfying the following condition: for every $a, b \in T$ all the points $v = (v_1, \ldots v_q)$ with the coordinates v_i equal either to a_i or b_i belong to T. We shall consider mainly processes with independent increments $\Delta X(V)$ on disjoint rectangles $V =< a, b) \subset R_+^q$, and for simplicity such processes will be called in short additive. Obviously, if the process in question is indexed by a proper subset $T \subset R_+^q$, then $\Delta X(V)$ is determined only for $a, b \in T$. The process X is said to be symmetric, if their finite dimensional distributions are symmetric (i.e. invariant under the inversion $x \to -x$) probability measures on finite products $(\mathbf{G}^L, \mathcal{G}^L)$ (or $(\mathbf{G}^L, \mathcal{G}(\mathbf{G}^L))$ provided group operations are ensured by $\mathcal{G}(\mathbf{G}^L)$-measurability). Note that under the above assumption the distribution of each increment of the process X is a symmetric probability measure. To derive our maximal symmetrization inequality instead of additive processes in fact a wider class of processes can be treated that is specified below.

Definition 3.1. We say that the process X_T has sign-invariant increments if for each finite array of disjoint rectangles $V_1, \ldots V_n \subset R$ and for arbitrary systems $\{S_1^{(i)}, \ldots S_{r_i}^{(i)}\}$ of subrectangles of V_i with endpoints in T, the random vectors

$$(\Delta X(S_1^{(1)}), \ldots \Delta X(S_{r_1}^{(1)}),\ \ldots\ \Delta X(S_1^{(n)}), \ldots \Delta X(S_{r_n}^{(n)})) \text{ and}$$
$$(\theta_1 \Delta X(S_1^{(1)}), \ldots \theta_1 \Delta X(S_{r_1}^{(1)}),\ \ldots\ \theta_n \Delta X(S_1^{(n)}), \ldots \theta_n \Delta X(S_{r_n}^{(n)}))$$

on $(\mathbf{G}^L, \mathcal{G}^L)$ (or $(\mathbf{G}^L, \mathcal{G}(\mathbf{G}^L))$ resp.), $L = r_1 + \ldots r_n$, have the same distribution for any choice of signs $\theta_i = \pm 1$.

Obviously, every additive symmetric process has sign-invariant increments on $(\mathbf{G}^L, \mathcal{G}^L)$, but the converse is not true.

Lemma 3.2. *Let X_T be a stochastic process with sign-invariant increments taking values in a T_0 topological Abelian group $\mathbf{G}$ with the σ-field $\mathcal{G}$. If $< w, z >$, $w, z \in T$ is a fixed bounded rectangle in R_+^q and $D \subset T$ is a finite set of points $t \in < w, z >$, then for every globular set U in $\mathbf{G}$ we have*

$$(3.1)\qquad P\Big[\bigcup_{s,t\in D}\Big(\Delta X(< s,t)) \notin U\Big)\Big] \le 4^q P\Big[\Delta X(< w,z)) \notin U(-2q)\Big].$$

In this article we omit the proof of the above inequality, for it is rather long and tedious. The proof of it will be presented elsewhere.

Corollary 3.3. *Let X_T be a stochastic process satisfying the hypotheses of Lemma 3.2 above and let $Q \subset T$ be a countable set of points $t \in < w, z >$. Then (3.1) remains true with D replaced by Q. Moreover, if in addition X_T is a separable process on $< w, z > \cap T$ with respect to closed sets $F \in \mathcal{G}$, then*

$$(3.2)\qquad P\Big[\bigcup_{s,t\in< w,z>\cap T}\Big(\Delta X(< s,t)) \notin U\Big)\Big] \le 4^q P\Big[\Delta X(< w,z)) \notin U(-2q)\Big].$$

We mention now some special cases of the above inequalities.

Corollary 3.4. *a). Let $X = \{X(t)\ ,\ t \in R_+^q = T\}$ be a symmetric additive stochastic process taking values in a linear topological Lindelöf space $\mathbf{G}$ with its Baire σ-field $\mathcal{G}$. Then for every open convex neighbourhood U of zero in $\mathbf{G}$ and arbitrary real number ε, $0 < \varepsilon < 1$,*

$$(3.3)\qquad P\Big[\bigcup_{s,t\in Q}\Big(\Delta X(< s,t)) \notin U\Big)\Big] \le 4^q P\Big[\Delta X(< w,z)) \notin \varepsilon U\Big],$$

where Q is a finite or countable subset of points of a bounded rectangle $< w, z > \subset R_+^q$. If in addition X is a separable process on $< w, z >$ with respect to closed sets $F \in \mathcal{G}$, then Q in (3.3) may be replaced by $< w, z >$. Moreover, letting $\varepsilon \nearrow 1$ through rational numbers we see that our inequality remains valid for $\varepsilon = 1$. In a special case when X is a Brownian sheet taking values in $\mathbf{R}$ and $U = (-\varepsilon, \varepsilon) \subset \mathbf{R}$, $\varepsilon > 0$, the obtained result reduces to Proposition 3.7 by Walsh [27]. Furthermore, if $\mathbf{G}$ is a separable Banach space and $q = 1$ we obtain a variant of inequality (18), Chapter IV, §1 in Gikhman and Skorohod [7].

b) Let $X = \{\xi(i), i \in \mathbf{N}^q = T\}$ be a stochastic process with sign-invariant increments taking values in a separable normed linear space $\mathbf{G}$ considered with the Borel σ-field $\mathcal{B} = \mathcal{G}$. Suppose that for a nondecreasing sequence $n = (n_1, \dots n_q) \in \mathbf{N}^q$, $n_1, \dots n_q \nearrow \infty$, $\|\sum_{i\le n} \xi(i)\|$ tends weakly to $\|S\|$, where S is a random element in $(\mathbf{G}, \mathcal{B})$. Then for every $\varepsilon > 0$,

$$(3.4)\qquad P\Big[\sup_{n\in\mathbf{N}^q}\Big\|\sum_{i\le n}\xi(i)\Big\| \ge \varepsilon\Big] \le 4^q P\Big[\|S\| \ge \varepsilon\Big]$$

and

$$(3.5) \qquad P\Big[\sup_{n\in\mathbb{N}^q}\|\xi(n)\|\geq\varepsilon\Big]\leq 4^q P\Big[\|S\|\geq\varepsilon\Big]\ .$$

For $q=1$ we obtain Lévy's inequalities (with constant 4 instead of 2) – see Proposition 2.3 and its Corollary, Chapter V, p. 210-211 in Vakhania, Tarieladze and Chobanian [28].

Remark. If the distributions of increments of the process X are Radon measures, then Corollary 3.4 a) remains true without Lindelöf property imposed on G. Indeed, in such a case for each rectangle $S\subset T$ there is an increasing sequence $K_1\subset K_2\subset\ldots\subseteq \mathbf{G}$ of compact sets such that $P[\Delta X(S)\in \mathbf{G}\setminus\bigcup_i K_i]=0$. Since every compact set is a Lindelöf space, we can choose at most denumerably many open Baire sets C_j so that $P[\Delta X(S)\in U^c\setminus\bigcup_j C_j]=0$, and this suffices for the proof of our inequality. Similarly, instead of assuming that G is separable in Corollary 3.4 b) we may consider separably valued random elements.

REFERENCES

[1] Adler, R. J., Monrad, D., Scissors, R. H., Wilson, R., *Representations, decompositions and sample function continuity of random fields with independent increments*, Stoch. Proc. Appl. 15 (1983), 3-30.

[2] Borell, C., *The Brunn-Minkowski inequality in Gauss space*, Inventiones Math. 30 (1975), 207-216.

[3] Borell, C., *Gaussian Radon measures on locally convex spaces*, Math. Scand. 38 (1976), 265-284.

[4] Doob, J. L., *Stochastic Processes*, Wiley, New York, Chapman & Hall, London 1953.

[5] Finetti, B., *Sulle funzioni a incremento aleatorio*, Rend. Acad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. (6) 10 (1929), 163-168.

[6] Gikhman, I. I., Skorohod, A. V., *Introduction to the Theory of Stochastic Processes*, Nauka, Moscow 1965 (in Russian).

[7] Gikhman, I. I., Skorohod, A. V., *The Theory of Stochastic Processes I*, Springer-Verlag, Berlin, Heidelberg 1973.

[8] Gnedenko, B. V., *On the growth of homogeneous stochastic processes with independent increments*, IAN SSSR, ser. mat. 7 (1943), 89-110 (in Russian).

[9] Gnedenko, B. V., *On the theory of growth of homogeneous stochastic processes with independent increments*, Sb. Trudov Inst. Mat. AN USSR 10 (1948), 60-82 (in Russian).

[10] Goodman, V., *Characteristics of normal samples*, Ann. Probab. 16 (1988), 1281-1290.

[11] Goodman, V., Kuelbs, J., *Rates of convergence for increments of Brownian motion*, J. Theoret. Probability 1 (1988), 27-63.

[12] Hewitt, E., Ross, K. A., *Abstract Harmonic Analysis*, Springer-Verlag, Berlin 1963.

[13] Heyer, H., *Probability Measures on Locally Compact Groups*, Springer-Verlag, Berlin, Heidelberg, New York 1977.

[14] Hoffmann-Jørgensen, J., Shepp, L. A., Dudley, R. M., *On the lower tail of Gaussian seminorms*, Ann. Probab. 7 (1977), 319-342.

[15] Itô, K., *On stochastic processes I. Infinitely divisible laws of probability*, Japan J. Math. 18 (1942), 261-301.

[16] J u r l e w i c z , T. , *Law of the iterated logarithm for Wiener processes with values in Orlicz spaces*, Probab. Math. Stat. 7 (1987), 159-167.

[17] K a t k a u s k a i t é , A. , *Random fields with independent increments*, Litovsk. Mat. Sbornik 12 (4) (1972), 75-85 (in Russian).

[18] K h i n t c h i n e , A. Ya. , *On local growth of stochastic processes without after-effect*, IAN SSSR, ser. mat. (1939), 487-508 (in Russian).

[19] K o l m o g o r o v , A. N. , *Sulla forma generale di un processo stocastico omogeneo*, Atti Acad. Naz. Lincei 15 (1932), 805-808, 866-869.

[20] K u e l b s , J. , L e P a g e , R. D. *The law of the iterated logarithm for Brownian motion in a Banach space*, Trans. Amer. Math. Soc. 185 (1973), 253-264.

[21] L e P a g e , R. D. , *Loglog law for Gaussian processes*, Z. Wahrsch. verw. Gebiete 25 (1972), 103-108.

[22] L é v y , P. , *Sur les intégrales dont les éléments sont des variables aléatoires indépendentes*, Ann. Scuola Norm. Pisa 2, No 3 (1934), 337-366.

[23] L o è v e , M. , *Probability Theory*, Van Nostrand, Princeton 1960.

[24] M u e l l e r , C. , *A unification of Strassen's law and Lévy's modulus of continuity*, Z. Wahrsch. verw. Gebiete 56 1981), 163-179.

[25] S k o r o h o d , A. V. , *Random Processes with Independent Increments*, Nauka, Moscow 1986 (in Russian).

[26] S t r a f , M. L. , *Weak convergence of stochastic processes with several parameters*, Proc. 6-th Berkeley Symp. Math. Statist. Probab. 2 (1972), 187-222.

[27] W a l s h , J. B. , *Martingales with a multidimensional parameter and stochastic integrals in the plane*, Lect. Notes in Math. 1215 , Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1986, 329-491.

[28] V a k h a n i a , N. N. , T a r i e l a d z e , V. I. , C h o b a n i a n , S. A. , *Probability Distributions in Banach Spaces*, Nauka, Moscow 1985 (in Russian).

[29] W i c h u r a , M. J. , *Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments*, Ann. Probab. 1 (1973), 272-296.

[30] Z a p a ł a , A. M. , *Regularisation of trajectories for multiparameter additive processes in groups*, J. Multivariate Anal. 36 (1989), 222-242.

[31] Z i n č e n k o , N. M. , *On the local growth of random fields with independent increments*, Teor. Veroyatn. Prim. XXIV, 1 (1979), 184-191 (in Russian).

STRESZCZENIE

W artykule wprowadzono pojęcie zbioru globularnego oraz opisano szeroką klasę grup mających globularne otoczenia zera. Następnie podane zostały różne kryteria na to, aby rodzina globularnych otoczeń zera była klasą górną lub dolną zbiorów dla przyrostów procesu addytywnego w grupie.

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