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The Boundary Correspondence under Quasiconformal Automorphisms of a Jordan Domain

Odpowiedniość brzegowa przy odwzorowaniach quasikonforemnych automorfizmów obszarów Jordana

Abstract. Let Γ be a Jordan curve in the extended plane $\overline{\mathbf{C}}$ and let D, D^* be its complementary domains. With every ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, two real values $[z_1, z_2, z_3, z_4]_D$ and $[z_1, z_2, z_3, z_4]_{D^*}$, are associated and called *conjugate harmonic cross-ratios*. Both of them are conformal invariants. Continuing his earlier work on the boundary value problem for quasiconformal automorphisms and using the above invariants, the author defines two classes $A_D(K)$ and $A_{D^*}(K)$ of automorphisms of Γ, and proves that they are representing the boundary values of all K-quasiconformal automorphisms of D and D^*, respectively. As an application, new characterizations of quasicircles are obtained.

1. Introduction. It is well-known that a K-quasiconformal (K-qc) automorphism F of a Jordan domain $D \subset \overline{\mathbf{C}}$, can be extended to a homeomorphism of the closure $\overline{D}$. It then induces an automorphism $f = F|_\Gamma$ of the boundary curve $\Gamma = \partial D$. In the case of $D = U = \{z : \operatorname{Im} z > 0\}$, and a K-qc automorphisms F of U that fixes the point at infinity, the induced automorphism $f = F|_{\mathbf{R}}$ of $\mathbf{R}$ can be represented by a ρ-quasisymmetric (ρ-qs) function in the sense of A. Beurling and L. V. Ahlfors (BA-condition) (see [3] and [10]). The family of all ρ-qs functions, $\rho \geq 1$, is invariant under composition only with increasing linear functions.

A characterization of the boundary values of K-qc automorphisms F of the unit disc $\Delta = \{z : |z| < 1\}$ was given by J. G. Krzyż (K-condition) in [6]. Using the conformal configuration connected with harmonic measure, he also obtained a class of ρ-qs functions of $T = \partial\Delta$, representing boundary automorphisms $f = F|_T$. This class of all ρ-qs functions, $\rho \geq 1$, is invariant under composition only with the group of rotations of T.

In both the cases, the ρ-qs functions have some deficiencies not shared by K-qc mappings (see [15]). In spite of extremal simplicity of these characterizations, it is not so easy to get a result asymptotically sharp for $\rho = 1$ (cf. [5], [4] and [7]). It is worthwhile to note that the BA-condition is not conformally transferable, whereas the K-condition is conformally invariant. The qs constant $\rho(f)$, defined as the minimum of all ρ such that the qs condition BA (or K) is satisfied by f, can not be used immediately to describe the Teichmüller distance without qc extensions.

Using the results of G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen ([1], [2], and other papers), as well as obtaining new ones (see [15]) on the Hersch–Pfluger distortion function Φ_K, the author was able to present a new characterization of the boundary values for the family of all K-qc automorphisms of a generalized disc in the extended complex plane $\overline{\mathbf{C}}$ (see [13] and [15]).

To describe this characterization let us recall that by a *generalized circle* (gc) $\Gamma \subset \overline{\mathbf{C}}$, we mean the stereographical projection of a circle on the Riemann sphere $\mathbf{B}^2 = \{(x,y,u) : x^2 + y^2 + u^2 - u = 0\}$. The following expression

$$[z_1, z_2, z_3, z_4] = \left\{ \frac{z_3 - z_2}{z_3 - z_1} : \frac{z_4 - z_2}{z_4 - z_1} \right\}^{1/2}, \tag{1.1}$$

introduced in [12], is well defined for each ordered quadruple of distinct points z_1, z_2, z_3, z_4 of a gc $\Gamma \subset \overline{\mathbf{C}}$. It is invariant under homographies and its values range over $(0;1)$.

By $A_\Gamma(K)$ we denote the family of all sense-preserving automorphisms f of a gc $\Gamma \subset \overline{\mathbf{C}}$, such that

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]) \leq [f(z_1), f(z_2), f(z_3), f(z_4)] \leq \Phi_K([z_1, z_2, z_3, z_4]) \tag{1.2}$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, with a constant $K \geq 1$.

A function $f \in A_\Gamma(K)$ is said to be the *K-quasihomography* (K-qh) of Γ. This class of functions represents the boundary values of all K-qc automorphisms of the domains D and D^*, complementary with respect to Γ, with the same constant K at the necessity. It is invariant under self-homographies of Γ, and has a number of properties close to those of K-qc mappings (see [15] and [16]). The relationships between K-qh and ρ-qs functions, in both the cases of $\Gamma = \overline{\mathbf{R}}$, or $\Gamma = T$, are obtained in [13] and [15]. Some fundamental results on $A_\Gamma(K)$ can be found in [16]. All of them are asymptotically sharp for $K = 1$. Nevertheless, the condition (1.2) is not conformally invariant.

Suppose that Γ is a Jordan curve (Jc) in $\overline{\mathbf{C}}$, while D and D^* are its complementary domains. Let $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$, be the classes of all K-qc automorphisms of D and D^*, respectively. If Γ is a gc of $\overline{\mathbf{C}}$, then $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$ are identical for each $K \geq 1$. In the case when Γ is a Q-quasicircle, $Q \geq 1$, both the classes are related by a Q^2-qc reflection in Γ, and can be identified on the level of the *universal Teichmüller space*, Theorem 11. In the most general case, when Γ is an arbitrary Jc of $\overline{\mathbf{C}}$, we do not have any quasiconformal relation between $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$, preserving points of Γ. This state of matter is an obstacle to our research on the boundary value problem for K-qc automorphisms. It means, that we can not simply start with a given Jc $\Gamma \subset \overline{\mathbf{C}}$, and a certain family of sense-preserving automorphisms of Γ, representing boundary values of $\mathcal{F}_D(K)$ or $\mathcal{F}_{D^*}(K)$.

The idea, that the starting point should be a Jc $\Gamma \subset \overline{\mathbf{C}}$, not a Jordan domain, when working with the boundary value problem for K-qc automorphisms, has its strong encouragement from the universal Teichmüller space theory (see [8, p. 97]). Keeping in mind this idea, we associate with a given Jc $\Gamma \subset \overline{\mathbf{C}}$, two classes $A_D(K)$

and $A_{D^*}(K)$ of sense-preserving automorphisms of Γ, representing the boundary values of $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$, respectively, with the same K at the necessity. In the case when Γ is a gc of $\overline{\mathbb{C}}$, the mentioned characterizations reduce to (1.2).

2. Conjugate harmonic cross-ratios. Let $\Gamma \subset \overline{\mathbb{C}}$ be an arbitrary Jc and let D, D^* be its complementary domains. Suppose that $a \in D$, is arbitrary and that $z', z'' \in \Gamma$, are arbitrary and distinct points of Γ. Consider

$$[z', z'']_D^a = \sin \pi\omega(a, <z', z''>; D), \tag{2.1}$$

where $<z', z''>$ is an oriented open arc of Γ, with end points z' and z'', ω being harmonic measure. It is obvious that $[z', z'']_D^a = [z'', z']_D^a$, where $<z'', z'> = \Gamma \setminus \overline{<z', z''>}$. Suppose that $z_1, z_2, z_3, z_4 \in \Gamma$, is an ordered quadruple of distinct points. Let

$$[z_1, z_2, z_3, z_4]_D^a = \{([z_2, z_3]_D^a [z_1, z_4]_D^a)/([z_1, z_3]_D^a [z_2, z_4]_D^a)\}^{1/2}. \tag{2.2}$$

Then we prove

Theorem 1. *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. For every $a, b \in D$, the identity*

$$[z_1, z_2, z_3, z_4]_D^a = [z_1, z_2, z_3, z_4]_D^b \tag{2.3}$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$.

Proof. Suppose that a and b are arbitrary points of D. By the Riemann mapping theorem, there are conformal mappings H_a and H_b, that map Δ onto D with $H_a(0) = a$ and $H_b(0) = b$. Both these mappings can be regarded as homeomorphisms of $\overline{\Delta}$ onto $\overline{D}$. By the conformal invariance of the harmonic measure, the equalities

$$\begin{aligned} [H_a^{-1}(z'), H_a^{-1}(z'')]_\Delta^o &= \sin \pi\omega(0, <H_a^{-1}(z'), H_a^{-1}(z'')>; \Delta) \\ &= \sin \pi\omega(H_a(0), <z', z''>; H(\Delta)) \\ &= \sin \pi\omega(a, <z', z''>, D) = [z', z'']_D^a \end{aligned} \tag{2.4}$$

hold for an arbitrary $z', z'' \in \Gamma$. The equality

$$[H_b^{-1}(z'), H_b^{-1}(z'')]_\Delta^o = [z', z'']_D^b \tag{2.4'}$$

holds by the same argument as (2.4). Let $z_1, z_2, z_3, z_4 \in \Gamma$, be an ordered quadruple of distinct points. Setting $t_k = H_a^{-1}(z_k)$ and $r_k = H_b^{-1}(z_k)$, $k = 1, 2, 3, 4$, then using (2.4) and (2.4′), it follows that

$$[z_1, z_2, z_3, z_4]_D^a = [t_1, t_2, t_3, t_4]_\Delta^o = [t_1, t_2, t_3, t_4], \tag{2.5}$$

and

$$[z_1, z_2, z_3, z_4]_D^b = [r_1, r_2, r_3, r_4]_\Delta^o = [r_1, r_2, r_3, r_4]. \tag{2.5'}$$

Since $H_b^{-1} \circ H_a$ is a conformal automorphism of Δ, then it is a homography mapping $\overline{\Delta}$ onto itself and thus it preserves (1.1). Therefore

$$(2.6)\ [r_1, r_2, r_3, r_4] = [H_b^{-1} \circ H_a(t_1), H_b^{-1} \circ H_a(t_2), H_b^{-1} \circ H_a(t_3), H_b^{-1} \circ H_a(t_4)] = [t_1, t_2, t_3, t_4].$$

This completes the proof.

Theorem 1 says that the expression, defined by (2.2), is a constant as a function of $a \in D$. By this we set

$$(2.7) \qquad [z_1, z_2, z_3, z_4]_D := [z_1, z_2, z_3, z_4]_D^a \qquad \text{for any } a \in D.$$

Note, that the statement of Theorem 1 remains true when we insert D^* instead of D, and $\Delta^* = \overline{\mathbf{C}} \setminus \Delta$ instead of Δ, respectively. Thus we define

$$(2.7') \qquad [z_1, z_2, z_3, z_4]_{D^*} := [z_1, z_2, z_3, z_4]_{D^*}^a \qquad \text{for any } a \in D^*.$$

Both these expressions, defined by (2.7) and (2.7′), are called the *conjugate harmonic cross-ratios* (*c.h. cross-ratios*).

Thus, with an arbitrary Jc $\Gamma \subset \overline{\mathbf{C}}$, and each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, we associate two values defined by (2.7) and (2.7′). The relationship between them will be of our special interest.

Theorem 2. *These c.h. cross-ratios are invariant under conformal mappings and their values range over* $(0;1)$ *for each ordered quadruple of distinct points* z_1, z_2, z_3, z_4 *of an arbitrary Jc* Γ *in* $\overline{\mathbf{C}}$. *Moreover,*

$$(2.8) \qquad [z_1, z_2, z_3, z_4]_D^2 = 1 - [z_2, z_3, z_4, z_1]_D^2,$$

where D *is one of the domains complementary with respect to* Γ.

These statements are obtained by the conformal invariance of *c.h. cross-ratios* and [12, Theorem 1]. Inserting D^* instead of D, we get the parallel result.

3. One dimensional qc mappings.

Suppose that Γ is an arbitrary Jc in $\overline{\mathbf{C}}$, where D and D^* are the domains complementary with respect to Γ.

Let A_Γ denotes the family of all sense-preserving automorphisms of Γ. This is evident that $(A_\Gamma, \circ)$ is a group with composition.

Definition 1. Let Γ be an arbitrary Jc in $\overline{\mathbf{C}}$, and let D, D^* be its complementary domains. An automorphism $f \in A_\Gamma$ is said to be of $A_D(K)$ class if

$$(3.1) \quad \Phi_{1/K}([z_1, z_2, z_3, z_4]_D) \le [f(z_1), f(z_2), f(z_3), f(z_4)]_D \le \Phi_K([z_1, z_2, z_3, z_4]_D)$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, and a constant $K \ge 1$.

The class $A_{D^*}(K)$ is defined by using D^* in (3.1) instead of D.

First we prove

Theorem 3. *Suppose that Γ is a Jc in $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. If $F \in \mathcal{F}_D(K)$ is an arbitrary, then $f = F|_\Gamma \in A_D(K)$ for each $K \geq 1$.*

Proof. Let H be a conformal mapping that maps Δ onto D. It can be regarded as a homeomorphism of $\overline{\Delta}$ onto $\overline{D}$. Let $F \in \mathcal{F}_D(K)$ be an arbitrary, where $K \geq 1$. The mapping

$$\tilde{F} = S_H(F) = H^{-1} \circ F \circ H \tag{3.2}$$

is a K-qc automorphism of Δ, and thus $\tilde{f} = \tilde{F}|_T \in A_T(K)$ (cf. [15] and [16]). Hence, by the conformal invariance of the *c.h. cross-ratios*, the proof of our theorem is established.

We may now describe the parallel theorem, whose statement is as follows: if $F \in \mathcal{F}_{D^*}(K)$ is an arbitrary then $f = F|_\Gamma \in A_{D^*}(K)$ for $K \geq 1$.

To show the sufficiency we prove

Theorem 4. *Suppose that Γ is a Jc in $\overline{\mathbb{C}}$, and that D and D^* are its complementary domains. For each $f \in A_D(K)$, $K \geq 1$, there exists a $K' = K'(K)$-qc automorphism F_f of D such that $F_{f|\Gamma} = f$.*

Proof. Let $f \in A_D(K)$, $K \geq 1$, be an arbitrary and let H be a conformal mapping of U onto D. Then $\tilde{f} = S_H(f)$ is an element of $A_{\overline{\mathbb{R}}}(K)$, and thus it has a $K' = K'(K)$-qc extension $F_{\tilde{f}}$ to U (cf. [15, Theorem 14]). By this

$$F_f = S_H^{-1}(F_{\tilde{f}}) \tag{3.3}$$

is the desired K'-qc automorphism of D, where $K' \leq \min\{\lambda^{3/2}(K), 2\lambda(K) - 1\}$ with $\lambda(K) = \Phi_K^2(1/\sqrt{2})/\Phi_{1/K}^2(1/\sqrt{2})$ (cf. [9]).

The parallel theorem for $f \in A_{D^*}(K)$, may be formulated automatically.

An automorphism $f \in A_D(K)$ (or $f \in A_{D^*}(K)$) is said to be a *1-dimensional K-qc (1-dim. K-qc) automorphism of* Γ. Both the classes $A_D(K)$ and $A_{D^*}(K)$, $K \geq 1$, are called *conjugate classes of 1-dim. K-qc automorphisms* of Γ. Let $f \in A_D(K)$, then the infimum $K_D(f)$, of all K such that (3.1) is satisfied, is said to be the *1-dim. qc constant of f*. Same we define $K_{D^*}(f)$ for $f \in A_{D^*}(K)$.

Some basic properties of *1-dim. K-qc* automorphisms are presented as:

Theorem 5. *For an arbitrary Jc $\Gamma \subset \overline{\mathbb{C}}$, and $K_1, K_2 \geq 1$, if $f_1 \in A_D(K_1)$ and $f_2 \in A_D(K_2)$, then $f_1 \circ f_2 \in A_D(K_1 K_2)$;*

Theorem 6. *For an arbitrary Jc $\Gamma \subset \overline{\mathbb{C}}$, and $K \geq 1$, if $f \in A_D(K)$, then $f^{-1} \in A_D(K)$.*

The proof of Theorem 5 follows immediately from the composition property of Φ_K and the definition of $A_D(K)$. Theorem 6 is a consequence of similar arguments. The parallel theorems may be formulated for $A_{D^*}(K)$.

Theorem 7. *Let Γ be an arbitrary Jc in $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. A function f is of $A_D(1)$ (or $A_{D^*}(1)$) class if, and only if, f is the boundary value of a conformal automorphism of D (or D^*).*

Proof. Let H maps conformally Δ onto D, and let $f \in A_D(1)$ be an arbitrary. The mapping $h = S_H(f) \in A_D(1)$ if, and only if, it is a homography mapping T onto itself (cf. [15, Theorem 11]. Denoting by H_* a conformal mapping of Δ^* onto D^*, then by the identity $A_T(1) = A_{\Delta^*}(1)$, we obtain the alternative assertion.

4. Quasicircles. Now we shall obtain the following characterizations of quasicircles as an application of the *c.h. cross-ratios* and the *conjugate* 1-*dim. K-qc* automorphisms of an arbitrary Jc $\Gamma \subset \overline{\mathbb{C}}$.

Theorem 8. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jc, and let D, D^* be its complementary domains. Then Γ is a quasicircle if, and only if, there exists a constant $K \geq 1$, such that*

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]_D) \leq [z_1, z_2, z_3, z_4]_{D^*} \leq \Phi_K([z_1, z_2, z_3, z_4]_D) \tag{4.1}$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$.

Proof. Suppose that Γ is a Q-quasicircle, $Q \geq 1$. Then there is a Q^2-qc reflection J_Γ in Γ. Let H and H_* be conformal mappings of Δ and Δ^*, onto D and D^*, respectively. The mapping

$$F = J_T \circ H_*^{-1} \circ J_\Gamma \circ H \tag{4.2}$$

is a qc automorphism of Δ. Consider $f = F|_T$ and an ordered quadruple of distinct points $w_1, w_2, w_3, w_4 \in T$. Then we have (cf. [15, Theorem 7])

$$\Phi_{1/Q^2}([w_1, w_2, w_3, w_4]) \leq [f(w_1), f(w_2), f(w_3), f(w_4)] \leq \Phi_{Q^2}([w_1, w_2, w_3, w_4]). \tag{4.3}$$

Due to the conformal invariance of the *c.h. cross-ratios*, it follows that

$$[w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4]_D,$$

where $w_i = H^{-1}(z_i)$, $i = 1, 2, 3, 4$. The reflection J_Γ does not change the points of Γ, whereas

$$[f(w_1), f(w_2), f(w_3), f(w_4)] = [z_1, z_2, z_3, z_4]_{D^*}$$

holds by the conformal invariance of the *c.h. cross-ratios*. Thus we have the necessity with $K = Q^2$.

($\Leftarrow$) Let Γ be a Jc in $\overline{\mathbb{C}}$, such that the inequalities (4.1) hold for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$. Consider $h = H^{-1} \circ H_*$ on T.

By (4.1) and by the conformal invariance of the *c.h. cross-ratios*, then the identity $[\cdot,\cdot,\cdot,\cdot]_\Delta = [\cdot,\cdot,\cdot,\cdot]_{\Delta^*}$, the following inequalities

$$(4.4)\quad \Phi_{1/K}([w_1,w_2,w_3,w_4]) \le [h(w_1),h(w_2),h(w_3),h(w_4)] \le \Phi_K([w_1,w_2,w_3,w_4])$$

hold for $w_i = H^{-1}(z_i)$. Hence, by [15, Theorem 14], there exists $K' = K'(K)$-qc automorphism F_h of Δ, with the boundary values given by h. Consider

$$(4.5)\quad G = H \circ J_T \circ F_h \circ H_*^{-1}.$$

We may see that G is a sense-reversing qc mapping of $\overline{D}$ onto $\overline{D^*}$, which is the identity on Γ. Defining $G(z) = G^{-1}(z)$ for $z \in D^*$, it follows that G is a K'-qc reflection in Γ, where $K' \le \min\{\lambda^{3/2}(K),\ 2\lambda(K)-1\}$ with $\lambda(K) = \Phi_K^2(1/\sqrt{2})/\Phi_{1/K}^2(1/\sqrt{2})$ (cf. Theorem 4), and consequently Γ is a quasicircle.

Let Γ be a Jc in $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. Denote by

$$(4.6)\quad R_\Gamma = \{f \in A_\Gamma : f = H \circ H_*^{-1}\},$$

where H and H_* are arbitrary conformal mappings of Δ and Δ^*, onto D and D^*, respectively. The family $S_H^{-1}(R_\Gamma) = CR(\Gamma)$ is said to be the *conformal representation of* Γ with respect to T (cf. [11]).

Let

$$(4.7)\quad A_D^\infty = \bigcup_{K\ge 1} A_D(K) \quad \text{and} \quad A_{D^*}^\infty = \bigcup_{K\ge 1} A_{D^*}(K).$$

Hence, $(A_D^\infty, \circ)$ and $(A_{D^*}^\infty, \circ)$ are subgroups of $(A_\Gamma, \circ)$, where $\circ$ denotes the composition.

The transformation

$$(4.8)\quad S_{HH_*} = S_{H_*}^{-1} \circ S_H$$

maps $A_D(K)$ onto $A_{D^*}(K)$ for every $K \ge 1$, and is an isomorphism between $(A_D^\infty, \circ)$ and $(A_{D^*}^\infty, \circ)$.

Since

$$(4.9)\quad S_{HH_*}(H \circ H_*^{-1}) = H \circ H_*^{-1},$$

then the *fix-points group* of S_{HH_*} contains the group $(R_\Gamma^\infty, \circ)$, generated by R_Γ (see [17]).

Hence, and by Theorem 6, the identities

$$(4.10)\quad K_D(H \circ H_*^{-1}) = K_D(H_* \circ H^{-1}) = K_{D^*}(H_* \circ H^{-1}) = K_{D^*}(H \circ H_*^{-1})$$

hold for every H and H_*, as above.

Definition 2. The common value, described by (4.10), we denote by K_Γ.

First we prove

Theorem 9. *If a Jc $\Gamma \subset \overline{\mathbf{C}}$, is a Q-quasicircle, $Q \geq 1$, then $R_\Gamma \in A_D(Q^2) \cap A_{D^*}(Q^2)$. Conversely, for each $K \geq 1$, there is $Q = Q(K)$ such that, if $R_\Gamma \in A_D(K) \cup A_{D^*}(K)$, then Γ is a $Q(K)$-quasicircle, where*
$1 \leq Q(K) \leq \min\{\lambda^{3/2}(K), 2\lambda(K) - 1\}$.

Proof. Suppose that $\Gamma \subset \overline{\mathbf{C}}$, is a Q-quasicircle, $Q \geq 1$. Then there is a Q^2-qc reflection in Γ. The mapping F, defined by (4.2), is a Q^2-qc automorphism of Δ. Thus $F|_T = H_*^{-1} \circ H \in A_T(Q^2)$. The automorphism

$$S_H^{-1}(H_*^{-1} \circ H) = S_{H_*}^{-1}(H_*^{-1} \circ H) = H \circ H_*^{-1} \tag{4.11}$$

is an element of $A_D(Q^2) \cap A_{D^*}(Q^2)$.

($\Leftarrow$) Suppose now that $H \circ H_*^{-1} \in A_D(K) \cup A_{D^*}(K)$, $K \geq 1$. The automorphism $H_*^{-1} \circ H \in A_\Delta(K) \cup A_{\Delta^*}(K) = A_T(K)$. Then, by [15, Theorem 14], there exists a $Q = Q(K)$-qc automorphism F_h of Δ, with the boundary values given by $h = H_*^{-1} \circ H$. From this moment we follow the sufficiency proof of Theorem 9, starting from (4.4), to obtain the sufficiency of this theorem.

Then we have

Theorem 10. *A Jc $\Gamma \subset \overline{\mathbf{C}}$, is a quasicircle if, and only if, $K_\Gamma < \infty$.*

Proof. It is an immediate consequence of the previous considerations and Theorem 9.

It is worth-while to note that a Jc $\Gamma \subset \overline{\mathbf{C}}$ is a gc in $\overline{\mathbf{C}}$ if, and only if, the identity

$$A_D(K) = A_{D^*}(K), \tag{4.12}$$

holds for each $K \geq 1$. Further, we have the following

Theorem 11. *If a Jc $\Gamma \subset \overline{\mathbf{C}}$ is a quasicircle, then*

$$A_D^\infty = A_{D^*}^\infty. \tag{4.13}$$

Proof. Suppose that $\Gamma \subset \overline{\mathbf{C}}$, is a Jc while H and H_* are conformal mappings of Δ and Δ^*, onto D and D^*, respectively. Assume that Γ is a Q-quasicircle, $Q \geq 1$, and that $f \in A_D^\infty$ is an arbitrary. Then there is $K \geq 1$, such that $f \in A_D(K)$.

Let

$$f_* = S_{HH_*}(f). \tag{4.14}$$

By the previous considerations, then Theorem 5 and Theorem 9, it follows that

$$K_{D^*}(f) \leq Q^4 K_D(f). \tag{4.15}$$

Hence, there is $1 \le L \le Q^4 K$, such that $f \in A_{D^*}(L)$. Starting with any $f \in A^{\infty}_{D^*}$, and using the fact that $S^{-1}_{HH_*} = S_{H_*H}$, we may obtain similar inclusion, by which the identity (4.13) follows.

Suppose now that Γ is an arbitrary Jc in $\overline{\mathbb{C}}$, for which the identity (4.13) holds, where D and D^*, denote the complementary domains.The author conjectures it sufficies to make Γ a quasicircle.

Let us note that Theorem 9 is a generalization of a result of J. G. Krzyż [6, Theorem 3], whereas Theorem 9 is close to a characterization obtained by D. Partyka [11, p. 13]. A continuation of this research, in the direction of the universal Teichmüller space theory, can be found in [17].

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STRESZCZENIE

Niech Γ będzie krzywą Jordana w płaszczyźnie domkniętej $\overline{\mathbb{C}}$ i niech D, D^* będą składowymi jej dopełnienia. Uporządkowanej czwórce punktów z_1, z_2, z_3, z_4 krzywej Γ można przyporządkować dwie liczby rzeczywiste $[z_1, z_2, z_3, z_4]_D$, $[z_1, z_2, z_3, z_4]_{D^*}$, które autor nazywa sprzężonymi dwustosunkami harmonicznymi. Są one konforemnie niezmiennicze. Kontynuując swe wcześniejsze prace na temat odpowiedniości brzegowej przy odwzorowaniach quasikonforemnych autor określa używając wprowadzonych przez siebie niezmienników dwie klasy $A_D(K)$, $A_{D^*}(K)$ automorfizmów Γ i wykazuje, że określają one wartości brzegowe wszystkich automorfizmów quasikonforemnych obszarów D i D^*. Jako zastosowanie podaje on nową charakteryzację quasiokręgów.

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