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Riemannian Vector Fields and Pontrjagin Numbers

Riemannowskie pola wektorowe i liczby Pontriagina

Римановы векторные поля и числа Понтрягина

In this note we shall give a partial generalization of the Bott theorem [2] on infinitesimal isometries and characteristic numbers to the so called Riemannian vector fields without singularities.

Let $\mathcal{F}$ be an oriented Riemannian foliation of dimension one on a compact, connected manifold M (cf. [5], [6]). Such a foliation is also called a Riemannian flow (cf. [3]) on M. Let $Q = TM/T\mathcal{F}$ be the normal (quotient) bundle and:

$$O \longrightarrow T\mathcal{F} \longrightarrow TM \xrightarrow{q} Q \longrightarrow O$$

the exact sequence of fibre bundles.

Definition. A vector field $\mathcal{R}X$ on M is said to be Riemannian if its orbits are leaves of certain Riemannian flow.

The Riemannian vector field gives a reduction $\mathcal{R}B^1(M)$ of the bundle $B^1(M)$ of linear frames on M to a subbundle with structural group consisting of matrices $\begin{bmatrix} a, & b \\ 0, & A \end{bmatrix}$ where $a \in R^*$, $b \in R^{n-1}$, $A \in GL(n-1, R)$. The frames of $\mathcal{R}B^1(M)$ are the adapted frames to $\mathcal{F}$, i.e. such ones that the first of them is tangent to $\mathcal{F}$.

To the bundle Q there corresponds the associated principal fibre bundle $B^1_T(M)$, the bundle of transversal frames. In [5] P. Molino has shown that this bundle admits the reduction to the subbundle of the orthonormal transversal frames E^1_T.

Let g_T be a transversal metric in Q which corresponds to this reduction and g a bundle-like metric in TM associated with g_T (cf. [6], [7]). Let $\mathcal{R}E^1(M)$ be a reduction of the fibre bundle $\mathcal{R}B^1(M)$ to a bundle with a structural group consisting of matrices $\begin{bmatrix} 1 & , & 0 \\ 0 & , & A \end{bmatrix}$ where $A \in O(n-1,R)$ and $E^1(M)$ the reduction of $B^1(M)$ to bundle with the structural group $O(n,R)$. We have:

$$\mathcal{R}E^1(M) \hookrightarrow E^1(M)\,.$$

Let $pr : R^n \to R^{n-1}$, $(x^0,x^1,\dots,x^{n-1}) \mapsto (x^1,\dots,x^{n-1})$. With each frame in $\mathcal{R}E^1(M)$ we can associate a frame in E^1_T by the formula :

$$\lambda(e_0,e_1,\dots,e_{n-1}) = (e_{1T},\dots,e_{n-1T})$$

where $q(e_i) = e_{iT}$, $i = 1,\dots,n-1$ and $e_0 \in T\mathcal{F}$. Conversely :

$$\lambda^{-1}(e_{1T},\dots,e_{n-1T}) = (e_0,e_1,\dots,e_{n-1})$$

where $(e_1,\dots,e_{n-1})$ are determined uniquely by the bundle-like metric g. A mapping:

$$\phi : \begin{bmatrix} 1 & , & 0 \\ 0 & , & A \end{bmatrix} \longleftrightarrow A$$

is an isomorphism of the groups. So we have proved:

Lemma 1. *The principal fibre bundles $\mathcal{R}E^1(M)$ and E^1_T are isomorphic.*

On the bundle E^1_T we have the canonical form $\Theta_T = (\Theta^1_T,\dots,\Theta^{n-1}_T)$ (cf. [5]) and on $\mathcal{R}E^1(M)$ we have the canonical form $\Theta = (\Theta^0,\Theta^1,\dots,\Theta^{n-1})$. These forms are connected by the relation:

$$\lambda^*\Theta_T = pr \circ \Theta \tag{$*$}$$

P. Molino in [5] defined a lifted foliation $\mathcal{F}^{(1)}$ on E^1_T starting with a Riemannian foliation $\mathcal{F}$ on M. Let $\mathcal{R}X^{(1)}$ be such lifting of the Riemannian vector field $\mathcal{R}X$ to E^1_T. Using the mapping λ we have therefore the vector field $\lambda^{-1}_*\mathcal{R}X$.

Lemma 2. $i_{\lambda^{-1}_*\mathcal{R}X}\Theta = (1,0,\dots,0)$.

Proof. $i_{\lambda^{-1}_*\mathcal{R}X}\Theta^i = 0$ for $i = 1,\dots,n-1$ follows from $(*)$ and the fact that Θ_T is the basic form for the lifted foliation $\mathcal{R}X^{(1)}$. $i_{\lambda^{-1}_*\mathcal{R}X}\Theta^0 = 1$ follows from the calculations in a local chart adapted to $\mathcal{R}X$.

It is proved in [5] that:

$$i_{\mathcal{R}X^{(1)}}\Omega_T = 0$$

where Ω_T is the curvature form on E^1_T. We know (cf. [1]) that:

$$\phi_* \circ \Omega = \lambda^*\Omega_T$$

where Ω is the curvature form in $\mathcal{R}E^1_T$, so we have:

$$i_{\mathcal{R}X(1)}\lambda^{*-1} \circ \phi_* \circ \Omega = i_{\mathcal{R}X(1)}\Omega_T = 0 .$$

Hence:

$$i_{\lambda_*^{-1}\mathcal{R}X(1)}\Omega = 0 .$$

From this fact and the formula

$$\Omega^i_j = \frac{1}{2}R^i_{jkl}\Theta^k \wedge \Theta^l$$

we obtain

Lemma 3. $\Omega \equiv 0 \mod (\Theta^1, \dots, \Theta^{n-1})$.

Theorem. *The Pontrjagin numbers of the manifold M* ($\dim M = n = 2k$) *are zeros.*

Proof. Using the Lemma 3 it is sufficient to write an invariant polynomial:

$$P_k(\Omega, \dots, \Omega) . \qquad (**)$$

Since in (**) the forms $\Theta^1, \dots, \Theta^{n-1}$ will recur at least two times so:

$$P_k(\Omega, \dots, \Omega) = 0 .$$

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STRESZCZENIE

W pracy pokazano, że znikanie liczb Pontriagina zwartej, zorientowanej i spójnej rozmaitości jest warunkiem koniecznym istnienia 1-wymiarowej foliacji riemannowskiej.

РЕЗЮМЕ

В данной работе доказывается, что обращение в нуль чисел Понтрягина компактного, ориентированного и связного многообразия - это необходимое условие существования одномерного, ориентированного риманово слоения.