ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL.XXXIX,13 SECTIO A 1985

Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

W.MOZGAWA

Curvature and Torsion Tensors of Quasi-connection on Manifold with Singular Tensor

Tensory krzywizny i skręcenia quasi-koneksji na rozmaitości z tensorem osobliwym

Тензоры кривизны и кручения квази-связности на многообразии с сингулярным тензором

Let $(FM, M, Gl(n), \omega)$ be a bundle of linear frames on M with a connection ω. It is well known that for any connection following structure equations hold:

$$\begin{aligned} d\Theta^\gamma &= \Theta^\alpha \wedge \omega_\alpha^\gamma + \tfrac{1}{2} T^\gamma_{\alpha\beta} \Theta^\alpha \wedge \Theta^\beta \\ d\omega_\mu^\lambda &= \omega_\mu^\gamma \wedge \omega_\gamma^\lambda + \tfrac{1}{2} R^\gamma_{\mu\alpha\beta} \Theta^\alpha \wedge \Theta^\beta \end{aligned} \tag{1}$$

where ω_α^γ, Θ^β, $\alpha, \beta, \gamma = 1, \dots, n$ are the connection form and canonical form on FM, resp.

Let's consider $n + n^2$ vector fields E_λ^μ, E_α on FM, dual to ω_α^γ and Θ^β. Usually we call these vector fields fundamental vector fields and standard vector fields, resp. We have the following identities for these vector fields :

$$\begin{aligned} &\Theta^\beta(E_\alpha) = \delta_\alpha^\beta \qquad \Theta^\beta(E_\alpha^\mu) = 0 \\ &\omega_\sigma^\rho(E_\alpha) = 0 \qquad \omega_\sigma^\rho(E_\lambda^\mu) = \delta_\lambda^\rho \delta_\sigma^\mu . \end{aligned} \tag{2}$$

We can write the structure equations (1) in the dual form :

$$\begin{aligned} [E_\lambda^\mu, E_\rho^\sigma] &= \delta_\rho^\mu E_\lambda^\sigma - \delta_\lambda^\sigma E_\rho^\mu , \qquad [E_\alpha, E_\lambda^\mu] = -\delta_\alpha^\mu E_\lambda \\ [E_\alpha, E_\beta] &= -T^\gamma_{\alpha\beta} E_\gamma - R^\lambda_{\mu\alpha\beta} E_\lambda^\mu \end{aligned} \tag{3}$$

Yung - Chow Wong has considered the natural question what is the set of n vector fields E_α on FM which satisfies the equation:

$$[E_\alpha, E^\beta_\lambda] = -\delta^\beta_\alpha E_\lambda.$$

In this way he obtained a generalization of the linear connection viz. the so called quasi-connection. The standard vector fields E_α of a quasi-connection are given locally by

$$E_\alpha = x^j_\alpha \left(C^i_j \frac{\partial}{\partial x^i} - x^k_\gamma \Phi^i_{jk} \frac{\partial}{\partial x^i_\gamma} \right)$$

where C^i_j, Φ^i_{jk} are functions of x^i only and such that on $U \cap U' \neq \emptyset$ with coordinate systems $(U, x^i), (U', x^{i'})$ we have

$$A^{a'}_j C^{i'}_{a'} = C^a_j A^{i'}_a$$

$$\Phi^a_{jk} A^{i'}_a = C^a_j A^{i'}_{ak} + A^{a'}_j A^{b'}_k \Phi^{i'}_{a'b'}$$

where

$$A^{i'}_a = \frac{\partial x^{i'}}{\partial x^a}, \qquad A^{i'}_{jk} = \frac{\partial^2 x^{i'}}{\partial x^j x^k}.$$

It is easy to see that, if the tensor C is non-singular on M, then

$$\Gamma^i_{jk} := C^{-1a}_j \Phi^i_{ak}$$

are components of a linear connection.

We assume that rank $C = m < n$ throughout this paper. We also assume that the distribution in C is involutive (i.e. there exist functions λ^i_{kl} such that $C^a_{[k} C^i_{l]|a} = \lambda^a_{kl} C^i_a$, cf. [6]).

In [6] Y - Ch. Wong has proved the following theorem :

If (C, Φ) is a quasi-connection on M, then for any tensors X, Y, Z of type $(1,0), (0,1), (1,1)$, respectively, on M

$$\begin{aligned} \nabla_l X^i &= C^a_l X^i_{|a} + X^a \Phi^i_{la} \\ \nabla_l Y_j &= C^a_l Y_{j|a} - \Phi^a_{lj} Y_a \\ \nabla_l Z^i_j &= C^a_l Z^i_{j|a} + Z^a_j \Phi^i_{la} - \Phi^a_{lj} Z^i_a \end{aligned} \tag{4}$$

are components in (U, x^i) of tensors of type $(1,1), (0,2), (1,2)$ respectively on M. Moreover, the following equations hold :

$$\begin{aligned} \nabla_l (X^i Y_j) &= (\nabla_l X^i) Y_j + X^i \nabla_l Y_j \\ \nabla_l (X^a Y_a) &= C^b_l (X^a Y_a)_{|b}. \end{aligned} \tag{5}$$

We call the operator ∇ the covariant derivative with respect to quasi-connection on M.

Having considered the third structure equation (3) Y-Ch.Wong established the following

Theorem (cf. [6]) *Let (C,Φ) be any quasi-connection on M. Assume that the tensor C is of constant rank m on M and its field of image m-planes is involutive, so that $C^a_{[k}C^i_{l]|a} = \lambda^a_{kl}C^i_a$ in every coordinate system (U,x^i). Then there exists on M a tensor S of type $(1,2)$ satisfying the equation:*

$$S^h_{kl}C^i_h = (\Phi^h_{[kl]} - \lambda^h_{kl})C^i_h \tag{6}$$

in every (U,x^i). Moreover, for any such tensor S

$$R^i_{jkl} = C^a_{[k}\Phi^i_{l]j|a} - \Phi^a_{[kj}\Phi^i_{l]a} - \Phi^a_{[kl]}\Phi^i_{aj} + S^a_{kl}\Phi^i_{aj} \tag{7}$$

are components in (U,x^i) of a tensor R of type $(1,3)$ on M.

However, this theorem is rather difficult for applications because the tensor S given in involved form is not unique. In this paper we give reasonable assumptions under which we are able to determine curvature and torsion tensor of quasi--connection. We also give the formulae of Levi-Cicita quasi-connection and some properties of above mentioned tensors.

We assume that C is a singular tensor of a quasi-connection (C,Φ) on M such that its Nijenhuis tensor

$$N(X,Y) = [CX,CY] - C[X,CY] - C[CX,Y] + C^2[X,Y] \tag{8}$$

is equal to zero. We hope that this assumption is reasonable because in the last time many structures with singular $(1,1)$ tensors were considered and the condition $N(X,Y)=0$ often appears in these papers.

Theorem 1. *If (C,Φ) is quasi-connection on M then*

$$T^a_{jk} = \Phi^a_{[jk]} + C^a_{[j|k]} + P^a_{kl} \tag{9}$$

are components in (U,x^i) of a tensor T of type $(1,2)$ on M where

$$P \in \{P \in TM \otimes \Lambda^2 TM^*;\ \operatorname{im} P = \ker C\}.$$

Proof. It is sufficient to consider the transformation law of $C^a_{[j|k]}$ and $\Phi^a_{[jk]}$ where the transformation law of $C^a_{[j|k]}$ is

$$C^a_{[j|k]}A^{i'}_a = -A^{i'}_{z|k}C^z_j + A^{i'}_{z|j}C^z_k + C^{i'}_{[a'|b']}A^{a'}_j A^{b'}_k.$$

Theorem 2. *If (C,Φ) is quasi-connection on M and the Nijenhuis tensor N of C is zero then the tensor T satisfies the identity (6)*

$$T^h_{kl}C^i_h = (\Phi^h_{[kl]} - \lambda^h_{kl})C^i_h$$

Proof. A local expressions of the Nijenhuis tensor N is

$$N^h_{kl} = C^a_{[k} C^h_{l]|a} - C^a_{[l|k]} C^h_a = 0 \, . \tag{10}$$

Let's rewrite the formula (6) in the form

$$S^h_{kl} C^i_h = \Phi^h_{[kl]} C^i_h - \lambda^h_{kl} C^i_h = \Phi^h_{[kl]} C^i_h - C^a_{[k} C^i_{l]|a} \, . \tag{11}$$

If we substitute (10) in (11) then

$$\begin{aligned} S^h_{kl} C^i_h &= \Phi^h_{[kl]} C^i_h - C^a_{[l|k]} C^i_a = \Phi^a_{[kl]} C^i_a + C^a_{[k|l]} C^i_a = \\ &= (\Phi^a_{[kl]} + C^a_{[k|l]}) C^i_a = (\Phi^a_{[kl]} + C^a_{[k|l]} + P^a_{kl}) C^i_a \, . \end{aligned}$$

so $T^a_{kl} = \Phi^a_{[kl]} + C^a_{[k|l]} + P^a_{kl}$ satisfies (6).
We shall call the tensor T the torsion tensor of quasi-connection.
Lemma 3. *The torsion tensor T can be globally defined by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, CY] - [CX, Y] + C[X,Y] + P(X,Y). \tag{12}$$

Proof. It is sufficient to consider (4).
Corollary 1. *The torsion tensor is skew-symmetric:*

$$T(X,Y) = -T(Y,X) \tag{13}$$

Corollary 2. *If T is the torsion tensor of quasi-connection (C,Φ) on M then (C,Ψ) where $\Psi = \Phi - \frac{1}{2}T$ is a new quasi-connection on M without torsion.*

Lemma 4. *The curvature tensor R can be globally defined by*

$$\begin{aligned} R_{XY} Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,CY]} Z - \\ &\quad - \nabla_{[CX,Y]} Z + \nabla_{C[X,Y]} Z + \nabla_{P(X,Y)} Z \, . \end{aligned} \tag{15}$$

Proof. It is sufficient to express (15) in local coordinates. For an arbitrary smooth function f on M we define:

$$\delta_Z f = Z^j C^k_j f_{|k} \, , \qquad Z = Z^j \frac{\partial}{\partial x^j} . \tag{16}$$

Let's introduce an exterior derivative δ of 1-forms with respect to the singular tensor C (with the condition $N = 0$) by

$$\begin{aligned} (\delta\omega)(X,Y) &= \delta_X \omega(Y) - \delta_Y \omega(X) + \\ &- \omega\left([X, CY] + [CX, Y] - C[X,Y] + P(X,Y)\right) . \end{aligned} \tag{17}$$

Now we can state the following

Theorem 5. *If (C,Φ) is quasi-connection on M then the following structure equations hold*

$$\begin{aligned} \delta(dx^i) + \omega^i_l \wedge dx^l &= \frac{1}{2}T^i_{jk}dx^j \wedge dx^k \\ \delta\omega^i_j + \omega^i_l \wedge \omega^l_j &= \frac{1}{2}R^i_{jkm}dx^k \wedge dx^m \end{aligned} \tag{18}$$

where $\omega^i_j = \Phi^i_{kj}dx^k$.

Proof is straightforward if one considers (18) and the definitions of T^i_{jk} and R^i_{jkm}.

Now we can say what does it mean that the quasi-connection is Riemannian.

Definition. The quasi-connection ∇ on M is said to be Riemannian with respect to scalar product g if

$$\delta_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\,, \qquad T(X,Y) = 0\,. \tag{19}$$

Theorem 6. *Let (M,g) be a Riemannian manifold. Then there exists a unique Riemannian quasi-connection for a given singular tensor C.*

Proof. We do it in the same way as in the classical proof. By summing up the identities:

$$\delta_U g(V,W) = g(\nabla_U V, W) + g(V, \nabla_U W)$$

$$\delta_V g(W,U) = g(\nabla_V W, U) + g(W, \nabla_V U)$$

$$-\delta_W g(U,V) = -g(\nabla_W U, V) - g(U, \nabla_W V)$$

we obtain with the help of (13)

$$\begin{aligned} 2g(\nabla_V U, W) &= \delta_U g(V,W) + \delta_V g(W,U) - \delta_W g(U,V) - \\ &- g(T(U,V),W) - g([U,CV] + [CU,V] - C[U,V], W) + \\ &+ g(T(W,U),V) + g([W,CU] + [CW,U] - C[W,U], V) + \\ &+ g(T(W,V),U) + g([W,CV] + [CW,V] - C[W,V], U)\,. \end{aligned} \tag{20}$$

It means that $\nabla_V U$ is completely determined by g, T, C and the derivatives of g and C. In the local coordinates (U, x^i) this quasi-connection has the following form:

$$\begin{aligned} \Phi^i_{kl} = \frac{1}{2}g^{is}\big(& C^a_l g_{ks|a} + C^a_k g_{ls|a} - C^a_s g_{kl|a} + \\ &+ g_{as}T^a_{lk} + g_{as}C^a_{[k|l]} + g_{ak}T^a_{sl} - g_{ak}C^a_{[l|s]} + \\ &+ g_{al}T^a_{sk} + g_{al}C^a_{[k|s]}\big)\,. \end{aligned} \tag{21}$$

It is necessary to check the transformation law of quasi-connection (C,Φ) given by (21) but the calculations are rather lengthy and are omitted here.

If we put $T = 0$ we obtain the quasi-connection which is a generalization of Levi-Civita connection

$$\Phi^i_{kl} = \frac{1}{2} g^{it} \left(C^a_l g_{kt|a} + C^a_k g_{tl|a} - C^a_t g_{kl|a} + \right. \qquad (22)$$
$$\left. + g_{at} C^a_{[k|l]} + g_{ak} C^a_{[l|t]} + g_{al} C^a_{[k|t]} \right) .$$

Corollary 8. *One can check directly that we have $\nabla g = 0$ for the above quasi-connection.*

Theorem 7. *For the curvature tensor of Riemannian manifold we have following identities:*

$$a)\quad R_{XY} Z = -R_{YX} Z$$
$$b)\quad R_{XY} Z + R_{YZ} X + R_{ZX} Y = 0$$
$$c)\quad g(R_{XY} Z, W) = -g(R_{XY} W, Z)$$
$$d)\quad g(R_{XY} Z, W) = g(R_{ZW} X, Y) .$$

Proof.

a) is straightforward

b) is almost straightforward but one should use few times Jacobi identity and our condition $N(X,Y) = 0$

c) under our assumption $N(X,Y) = 0$ we have

$$\delta_{[X,CY]+[CX,Y]-C[X,Y]+P(X,Y)} g(W,Z) = (\delta_X \delta_Y - \delta_Y \delta_X) g(W,Z) \qquad (24)$$

and now the proof is similar to the classical case.

d) it follows from a), b), and c).

Taking into account the parallel displacement by D i C o m i t e [1] we will give a geometric interpretation of the curvature tensor of quasi-connection. According to [1] the parallel displacement of a vector field X holds along an integral curve of a vector field $C(Y)$ and is given by

$$\nabla_Y X = \left(\frac{dX^k}{dt} + Y^i \Phi^k_{ij} X^j \right) \frac{\partial}{\partial x^k} = 0 \qquad (25)$$

We shall move infinitesimally a frame $A^i_j \frac{\partial}{\partial x^i}$ (such that $A^i_j(m_0) = \delta^i_j$) starting in the direction of the vector Y and then in the direction of the vector X,afterwards we shall subtract from this a quantity obtained by theparallel displacement of the frame $A^i_j \frac{\partial}{\partial x^i}$ at first inthe direction X and then in the direction Y.

It is well known that in the case of linear connection we shall obtain an infinitesimal of second order that is just a curvature tensor, here we will obtain also a curvature tensor but with a "correction".

Let's now consider the equation

$$\nabla_X A^i_j = 0 \tag{26}$$

that is

$$\frac{dA^i_j}{dt} + \Phi^i_{kt} X^k A^t_j = 0 \tag{27}$$

hence

$$A^i_j(m_0 + tX + h.o.t.) = \delta^i_j - t\Phi^i_{kj}(m_0) X^k(m_0) + h.o.t. \tag{28}$$

where h.o.t. denotes the higher order terms. Now we shall move the frame $A^i_j(m_0 + tX + h.o.t.)$ in the direction of the vector Y starting at the point $m_0 + tX + h.o.t.$

$$\begin{aligned} A^i_j(m_0 + tX + tY + h.o.t.) &= \left(\delta^i_j - t\Phi^i_{kj}(m_0)X^k(m_0) + h.o.t.\right) - \\ &-t\Phi^i_{kp}\left(m_0 + tC(X) + h.o.t.\right) Y^k \left(m_0 + tC(X) + h.o.t.\right) \\ &\left(\delta^p_j - t\Phi^p_{zj}(m_0)X^z(m_0) + h.o.t.\right) + h.o.t. = \\ &= \delta^i_j - t\left(\Phi^i_{kj}X^k + \Phi^i_{kj}Y^k\right)_{|m_0} - t^2\left(\Phi^i_{kj|a}C^a_l X^l Y^k + \right. \\ &\left. + \Phi^i_{kj}\delta_{C(X)}Y^k - \Phi^i_{kp}\Phi^p_{zj}X^z Y^k\right)_{|m} + h.o.t. \end{aligned} \tag{29}$$

If we perform it in opposite order then we obtain

$$\begin{aligned} \tilde{A}^i_j(m_0 + tX + tY + h.o.t.) &= \delta^i_j - t(\Phi^i_{kj}Y^k + \Phi^i_{kj}X^k)_{|m_0} + \\ &-t^2(\Phi^i_{kj}\delta_{C(Y)}X^k + \Phi^i_{kj|a}C^a_l Y^l X^k - \Phi^i_{kp}\Phi^p_{zj}X^k Y^z)_{|m_0} + h.o.t.\,. \end{aligned} \tag{30}$$

After subtraction we obtain

$$\begin{aligned} A^i_j - \tilde{A}^i_j &= t^2[R^i_{jkl}X^l Y^k + \Phi^i_{kj}([CX,Y]^k + \\ &+[X,CY]^k - C[X,Y]^k - P^k(X,Y)] + h.o.t.\,. \end{aligned} \tag{31}$$

Thus the term at t^2 is just the curvature tensor of quasi-connection and the "correction"

$$[CX,Y] + [X,CY] - C[X,Y] + P(X,Y] \tag{32}$$

which also appeared in (12), (15), and (17). The correction (32) has the following interpretation – it is a counterpart of the Poisson bracket for the vector fields X and Y with respect to the singular tensor C.

REFERENCES

[1] D i C o m i t e, C., *Pseudoconnessioni lineari su una varietà differenziabile di classe* C^{∞}, Ann. Mat. Pura Appl., 83 (1969), 133–152.

[2] G i u b l e s i, D., *Connessioni affini generalizzate su varietà differenziabile e loro proprietà*, Univ. e Politec. Torino Rend. Sem.Mat., 29 (1969/70), 297–314.

[3] M o z g a w a, W., *Quasi-connections in the semiholonomic frame bundle of second order and their differential invariants*, An.Stiint.Univ."Al.I.Cuza" Iaşi Secţ. I a Mat., 27 (1981), 297–316.

[4] S p e s i v y h, V. L., *A generalized connection in a vector bundle* (Russian), Ukrain.Mat.Z., 30 (1978), 686–689.

[5] V a m a n u, E., *Quasi-connections on the differentiable manifolds* (Rumanian), An.Ştiinţ. Univ. "Al.I.Cuza" Iaşi Secţ. I a Mat., 16 (1970), 353–388.

[6] W o n g, Y - C h., *Linear connections and quasi-connections on a differentiable manifold*, Tôhoku Math.J., 14 (1962), 48–63.

STRESZCZENIE

Dla quasi-koneksji spełniającej pewne naturalne założenia wyznaczono tensory skręcenia i krzywizny oraz podano ich własności. Podano także uogólnienie koneksji Levi-Civita.

РЕЗЮМЕ

Для квази-связности выполняющей некоторые естественные условия получено тензоры кручения и кривизны вместе с тем представлено их свойства. Получено обобщение связности Леви-Чивита.