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On a Problem of S.S Miller

O pewnym problemie S.S Millera

О некоторой проблеме С.С Милера

Sanford S. Miller ([1], p.554) posed the following problem : Let $w(z)$ be a function regular in $E = \{z \in C : |z| < 1\}$ such that $w(0) = 0$. Does the condition :

$$|w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| < 1 , \quad z \in E , \tag{1}$$

imply that $|w(z)| < 1$ for $z \in E$?

S.S.Miller and P.T.Mocanu [5] have shown that this implication holds for $n = 2$. This paper contains the affirmative answer to this question in the general case. Moreover, the bound for $|w(z)|$ in the general case obtained here is sharp and equal $\frac{1}{2}$ for $n = 1, 2, \ldots$. The author is grateful to Prof. J.G. Krzyż for his encouragement and advice.

Miller's problem is related to Euler's differential equation:

$$z^n w^{(n)} + \cdots + zw' + w = f(z). \tag{2}$$

It is easily seen that for any function $f(z)$ regular in E there exists a unique solution of (2) regular in E which can be obtained by comparison of Taylor coefficients of both sides.

Let **B** denote the family of functions regular in E of the form:

$$g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots \tag{3}$$

which satisfy $|g(z)| < 1$ for $z \in E$.

The above problem may be reformulated in the following, equivalent way. Is it true that the holomorphic solution of (2) belongs to the class **B** if right hand side of (2) belongs to this class?

We shall need the following :

Lemma 1. *We have:*

$$S \stackrel{\text{def}}{=} \sum_{k=1}^{\infty}(1+k^2)^{-2} = \frac{1}{4}\left[\pi \coth \pi + \left(\frac{\pi}{\sinh \pi}\right)^2\right] - \frac{1}{2} = 0.3068\ldots .$$

Proof. The function $f(z) = (1+z^2)^{-2}$ is meromorphic in the finite plane C and has poles i, $-i$ none of which is an integer. Moreover, $\lim_{z\to\infty} zf(z) = 0$. These properties of the function $f(z)$ imply that the limit $\lim_{N\to+\infty} \sum_{k=-N}^{N} f(k)$ exists and equals to $-\operatorname{res}(i, \pi f(z)\cot \pi z) - \operatorname{res}(-i, \pi f(z) \cot \pi z)$. The proof can be completed by finding these residues (cf. [4], p. 69).

If $g \in \mathrm{B}$ then obviously $\sum_{k=1}^{\infty} |b_k|^2 \leq 1$. We shall prove Miller's conjecture under this weaker assumption. Then we have:

Theorem 1. *If $w(z)$ is regular in E, $w(0) = 0$, and (2) holds for some $n \geq 2$ and all $z \in E$ with $f(z) = \sum_{k=1}^{\infty} b_k^{(n)} z^k$ such that $\sum_{k=1}^{\infty} |b_k^{(n)}|^2 \leq 1$ then:*

$$|w(z)| \leq S^{\frac{1}{2}} = 0.5539\ldots . \tag{4}$$

The bound is sharp for $n = 2$ and is attained for:

$$w(z) = S^{-\frac{1}{2}} \sum_{k=1}^{\infty}(1+k^2)^{-2} z^k , \tag{5}$$

$$f(z) = S^{-\frac{1}{2}} \sum_{k=1}^{\infty}(1+k^2)^{-1} z^k . \tag{6}$$

Proof. Suppose that $n \geq 2$ and:

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots , \qquad z \in E \tag{7}$$

$$f(z) = w(z) + zw'(z) + \cdots + z^n w^{(n)}(z) = b_1^{(n)} z + b_2^{(n)} z^2 + \cdots . \tag{8}$$

Hence, for $k \leq n$ we have:

$$b_k^{(n)} = c_k[1 + k + k(k-1) + k(k-1)(k-2) + \cdots + k!];$$

if $k > n$, then :

$$b_k^{(n)} = c_k[1 + k + k(k-1) + \cdots + k(k-1)(k-2)\cdots(k-n+1)].$$

Thus in both cases:

$$|b_k^{(n)}| \geq |c_k|(1 + k + k^2 - k) = (1 + k^2)|c_k|, \qquad k = 2,3,\dots . \tag{9}$$

If $n = 2$ then the above inequality becomes to an equality. Now we will find an estimate for the modulus of $w(z)$. We have:

$$|w(z)| \leq \sum_{k=1}^{\infty} |c_k| \leq \sum_{k=1}^{\infty} (1 + k^2)^{-1} |b_k^{(n)}|, \qquad z \in E. \tag{10}$$

Using the Schwarz-Cauchy inequality we obtain :

$$|w(z)| \leq \sqrt{\sum_{k=1}^{\infty} |b_k^{(n)}|^2} \sqrt{\sum_{k=1}^{\infty} (1 + k^2)^{-2}} \leq \sqrt{S}, \qquad z \in E. \tag{11}$$

The example of functions (5) and (6) shows that the estimate (4) is sharp and this ends the proof of Theorem 1.

The estimate (4) is sharp in a wider class then **B**, but is not sharp in **B**, because the extremal functions (6) does not belong to **B**. Now, we shall give a detailed solution of the S.S Miller's original problem. First we prove:

Theorem 2. *If $w(z)$ is a function regular in E such that $w(0) = 0$ and $|w(z) + zw'(z)| \leq 1$ for $z \in E$ then $|w(z)| < \frac{1}{2}$. This estimate is sharp.*

Proof. Put:

$$zw'(z) + w(z) = [zw(z)]' = g(z),$$

where $g \in \mathbf{B}$. Then we have:

$$zw(z) = \int_0^z g(t)dt$$

and

$$|zw(z)| = \left|\int_0^1 g(tz)zdt\right| \leq \int_0^1 |tz||z|dt = \frac{1}{2}|z|^2,$$

The last inequality implies $|w(z)| < \frac{1}{2}z$, $z \in E$. Consider now $w(z) = \frac{1}{2}$. Then $|w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| = |z| < 1$ in E and $|w(z)| < \frac{1}{2}$ in E. This shows that the constant $\frac{1}{2}$ is best possible in case $n = 1$. We shall prove that it is so for $n \geq 1$ as well.

To this end we need following lemmas. A sequence (t_n) of nonnegative numbers is called a convex null sequence if $t_n \to 0$ as $n \to \infty$ and

$$t_{k-1} - t_k \geq t_k - t_{k+1}, \quad k = 2, 3, \ldots . \tag{12}$$

Lemma 2. *If $s_n = t_n^{-1}$ satisfies:*

$$s_k \geq (s_{k-1} s_{k+1})^{1/2}, \qquad k = 2, 3, \ldots \tag{13}$$

and $\lim s_n = +\infty$ *then* (t_n) *is a convex null sequence.*

Proof. From (13) and the well-known inequality between the geometric and harmonic mean we obtain:

$$t_k^{-1} = s_k \geq (s_{k-1} s_{k+1}^{1/2}) \geq \frac{2}{(s_{k-1}^{-1} + s_{k+1}^{-1})} = \frac{2}{t_{k-1} + t_{k+1}}$$

which is equivalent to (12).

Note that the converse is false. The sequence ($\exp(-n^2)$) is a convex null sequence whereas ($\exp(n^2)$), does not satisfy (13).

Lemma 3. ([3], p. 103). *Suppose (t_n), $t_1 > 0$ is a convex null sequence. Then:*

$$p(z) = \frac{t_1}{2} + \sum_{k=2}^{\infty} t_k z^{k-1}$$

satisfies $\mathrm{Re}\,(p(z)) > 0$ *for* $z \in E$.

Lemma 4. *Let $h_1 = 1$, h_2, h_3, ... be complex numbers, let $g \in \mathbf{B}$ satisfy* (3) *and put $T(g)(z) = \sum_{k=1}^{\infty} h_k b_k z^{k-1}$. Then $T(g) \in \mathbf{B}$ for all $g \in \mathbf{B}$ iff*

$$\mathrm{Re}\left(1 + 2\sum_{k=2}^{\infty} h_k z^{k-1}\right) > 0 \text{ for } z \in E.$$

This Lemma is a slight modification of a theorem due to Goluzin (cf.[2], p. 493). We now prove :

Theorem 3. *If $w(z)$ is regular in E, $w(0) = 0$ and* (2) *holds for some $n \geq 2$ and $z \in E$ with f belonging to the class* $\mathbf{B}$, *then $|w(z)| < \frac{1}{2}$. The bound is sharp for all n and is attained for $w(z) \equiv \frac{1}{2} z$ and $f(z) \equiv z$.*

Proof. As in Theorem 1 we put ;

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \qquad z \in E . \tag{14}$$

$$f(z) = w(z) + z w'(z) + \cdots + z^n w^{(n)}(z) = b_1^{(n)} z + b_2^{(n)} z^2 + \cdots . \tag{15}$$

We have:

$$b_k^{(n)} = c_k [1 + k + k(k-1) + \cdots + k(k-1)(k-2)\cdots(k-n+1)] . \tag{16}$$

Let us denote:

$$t_k^{(n)} = [1 + k + k(k-1) + \cdots + k(k-1)(k-2)\cdots(k-n+1)]^{-1} . \qquad (17)$$

If k is fixed then the sequence $t_k^{(n)}$ is weakly decreassing and $t_1^{(n)} = \frac{1}{2}$ for $n \geq 2$. By the formulas (14)-(17) we have:

$$2|w(z)| = \left|\sum_{k=1}^{\infty} 2t_k^{(n)} b_k^{(n)} z^{k-1}\right| = |T(f)(z)| . \qquad (18)$$

Let us consider the function:

$$p(z) = \frac{1}{4} + \sum_{k=2}^{\infty} t_k^{(n)} z^{k-1} \qquad (19)$$

We shall prove that for $n \geq 2$:

$$\mathrm{Re}\,(p(z)) > 0 \qquad \text{for } z \in E . \qquad (20)$$

If $n = 2$, then $t_k^{(2)} = (1 + k^2)^{-1}$. Let us put $s_k = (1 + k^2)$, $k = 1, 2, \ldots$. The sequence (s_k) satisfies the conditions of Lemma 2. By Lemmas 2 and 3 we obtain (20).

If $n = 3$ then $t_k^{(3)} = (k^3 - 2k^2 + 2k + 1)^{-1}$. Let us put $s_k = [t_k^{(3)}]^{-1}$, $k = 1, 2, \ldots$. The sequence (s_k) satisfies (13) for $k \geq 3$ and $t_1^{(3)} - t_2^{(3)} \geq t_2^{(3)} - t_3^{(3)}$. The above condition implies that sequence $(t_k^{(3)})$ is a convex null sequence. From Lemma 3 we obtain (20).

If $n \geq 4$ then

$$p(z) = \frac{1}{4} + \sum_{k=2}^{\infty} t_k^{(3)} z^{k-1} + \sum_{k=2}^{\infty} [t_k^{(n)} - t_k^{(3)}] z^{k-1} . \qquad (21)$$

By Lemma 3

$$\mathrm{Re}\left(\frac{3}{16} + \sum_{k=2}^{\infty} t_k^{(3)} z^{k-1}\right) > 0 \qquad \text{for } z \in E . \qquad (22)$$

Now we will estimate a remaining term in (21):

$$\left|\sum_{k=2}^{\infty} (t_k^{(n)} - t_k^{(3)}) z^{k-1}\right| \leq \sum_{k=4}^{\infty} t_k^{(4)} < \sum_{k=4}^{\infty} k^{-3} \approx 0.39 < \frac{1}{16} \qquad \text{for } z \in E . \qquad (23)$$

From (22) and (23) we obtain $\mathrm{Re}\, p(z) > 0$ for $z \in E$. From the equality (20) we have:

$$\mathrm{Re}\,(4p(z)) = \mathrm{Re}\left(1 + \sum_{k=2}^{\infty} 4t_k^{(n)} z^{k-1}\right) > 0 \qquad \text{for } z \in E . \qquad (24)$$

Using Lemma 4 with $h_k = 2t_k^{(n)}$ and taking into account (24), (18) we verify that $T(f) \in \mathbf{B}$ and this gives an estimate $|w(z)| < \frac{1}{2}$ for $z \in E$. The example $w(z) \equiv \frac{1}{2}z$, $f(z) \equiv z$ shows that this estimate is sharp.

REFERENCES

[1] Brannan, D.A., Clunie, J.G., *Aspects of Contemporary Complex Analysis*, Academic Press, New York-San Francisco 1980.

[2] Goluzin, G., *Geometric theory of functions of a complex variable*, (Russian), GITTL, Moscow 1966.

[3] Goodman A. W., *Univalent Functions*, Mariner Publishing Company, Tampa 1983.

[4] Krzyż, J. G., *Problems in Complex Variable Theory*, American Elsewier Publ. Co., New York 1971.

[5] Miller, S.S., Mocanu, P.T., *Second order differential inequalities in the complex plane*, J.Math.Anal.Appl. 65 (1978), 289–305.

STRESZCZENIE

Praca dotyczy następującego problemu Millera ([1], str.554): "Niech $w(z)$ będzie funkcją regularną w $E = \{z \in C : |z| < 1\}$ taką, że $w(0) = 0$. Czy warunek

$$|w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| < 1\,, \qquad z \in E$$

implikuje zależność $|w(z)| < 1$ dla $z \in E$?"

W pracy wykazano, że z (1) wynika mocniejszy warunek : $|w(z)| < \frac{1}{2}$ dla $z \in E$.

РЕЗЮМЕ

В данной работе рассматривается следующая проблема Милера ([1], стр.554): "Пусть $w(z)$ обозначает функцию регулярную в $E = \{z \in C : |z| < 1\}$ такую, что $w(0) = 0$. Будет ли верно, что условие :

$$|w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| < 1, \quad z \in E\,,$$

влечет зависимость $|w(z)| < 1$ для $z \in E$?" В работе показано, что из (1) вытекает сильнейшее условие : $|w(z)| < \frac{1}{2}$ для всех $z \in E$.