ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XXXIX, 8 SECTIO A 1985

Department of Mathematical Sciences
University of Delaware Newark , Delaware

D. J. HALLENBECK

A Note on the Riesz – Herglotz Representation

O reprezentacji Riesza – Herglotza

О представлении Рисса – Герглотца

Introduction. Let $\Delta = \{z : |z| < 1\}$ and let **A** denote the set of functions analytic in Δ. Then **A** is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ . A function f is called a support point of a compact subset **F** of **A** if $f \in \mathbf{F}$ and if there is a continuous, linear functional J on **A** so that Re $J(f) = \max\{\mathrm{Re} J(g) : g \in \mathbf{F}\}$ and Re J is nonconstant on **F**. We denote the set of support points of such a family by supp **F** and the closed convex hull of such a family we denote by $H\mathbf{F}$. Since $H\mathbf{F}$ is itself compact, the set of extreme points of $H\mathbf{F}$ which we denote by $EH\mathbf{F}$ is non-void.

A function $f \in \mathbf{A}$ is said to be subordinate to a function $\mathbf{F} \in \mathbf{A}$ if there exist $\phi \in \mathbf{A}$ such that $\phi(0) = 0, |\phi(z)| < 1$ and $f = \mathbf{F} \circ \phi$. We let B_0 denote the set of functions $\phi \in \mathbf{A}$ and satisfying $|\phi(z)| \leq |z| (|z| < 1)$. The set of functions subordinate to **F** we denote by $s(\mathbf{F})$ and note that $s(\mathbf{F}) = \{\mathbf{F} \circ \phi : \phi \in \mathrm{B}\}$. It is known that supp B_0 consists of all finite Blaschke products which vanish at the origin [1],[4]. If $\mathbf{F} \in \mathbf{A}$ it was proved in [2] that supp $s(\mathbf{F}) \subseteq \{\mathbf{F} \circ \phi : \phi \in$ supp $\mathrm{B}_0\}$. This inclusion was proved in [6] under the additional assumption that $\mathbf{F}'(z) \neq 0$ for $z \in \Delta$.

In recent years a number of proofs of the Riesz-Herglotz representation for $s\left(\frac{1+z}{1-z}\right)$ have been given [5],[8]. The basis of these arguments has been a proof that $Es\left(\frac{1+z}{1-z}\right) = \left\{\frac{1+xz}{1-xz} : |x| = 1\right\}$. The desired representation formula then follows by appeal to Choquet's theorem [9] or to the Krein-Milman theorem and the weak star compactness of the set of probability measures on $\partial\Delta$.

In this short note we give a new proof of the Riesz-Herglotz representation. The proof uses the knowledge of supp B_0 mentioned above [1],[4]. It also depends on the observation made in [7,p.92] that H **F** $= H($ supp **F** $\cap EH$**F**$)$ for any compact family **F** contained in **A**. We note that $s\left(\frac{1+z}{1-z}\right) = Hs\left(\frac{1+z}{1-z}\right)$ since $\frac{1+z}{1-z}$ is convex and univalent. Also, in [6] the set supp $s\left(\frac{1+z}{1-z}\right)$ was exactly determined. We do not use this result since its proof depended in part on knowning the Riesz – Herglotz representation.

We also give a new proof a generalization of the Riesz – Herglotz formula that was proved in [3] by D.A.Brannan, J.G.Clunie and W.E.Kirwan.

The Riesz-Herglotz representation. Theorem. *A function* $p \in s\left(\frac{1+z}{1-z}\right)$ *if and only if there is a probability measure* μ *on* $\partial\Delta$ *such that*

$$p(z) = \int_{|x=1|} \frac{1+xz}{1-xz} d\mu(x) \qquad (|z|<1)\,. \tag{1}$$

Proof. It is clear that each function p of the form (1) is in $s\left(\frac{1+z}{1-z}\right)$ since $\frac{1+z}{1-z}$ is univalent and convex in Δ and Re $p(z) \geq 0$, $p(0) = 1$.

We now prove that each $p \in s\left(\frac{1+z}{1-z}\right)$ has the form (1). It is known that supp $s\left(\frac{1+z}{1-z}\right) \subseteq \left\{\frac{1+\phi}{1-\phi} : \phi \in \text{ supp } B_0\right\}$ [2], [6]. It follows from the fact that $\phi \in$ supp B_0 and from lemma 4 in [4,p.82] that

$$\frac{1+\phi(z)}{1-\phi(z)} = \sum_{k=1}^{n} \lambda_k \frac{1+x_k z}{1-x_k z} \qquad (|z|<1) \tag{2}$$

where $0 \leq \lambda_k \leq 1$, $\sum_{k=1}^{n} \lambda_k = 1$ and $|x_k| = 1$ $(k = 1,2,\ldots,n)$. Also, $\lambda_k = 1$ for some k if and only if $\phi(z) = xz$ for some $|x| = 0$. It follows from (2) that if $\phi \in$ supp B_0 and $\phi(z) \neq xz$ for some $|x| = 1$ then $\frac{1+\phi}{1-\phi} \notin Es\left(\frac{1+z}{1-z}\right)$. Hence we deduced that

$$\text{supp } s\left(\frac{1+z}{1-z}\right) \cap Es\left(\frac{1+z}{1-z}\right) \subseteq \left\{\frac{1+xz}{1-xz} : |x| = 1\right\}.$$

However, it is known in general that $\{F(xz) : |x = 1\}$ is contained in $EHs(F)$ and supp $s(F)$ for any nonconstant F in **A** [7,p.50p.103]. So it follows that

$$\text{supp } s\left(\frac{1+z}{1-z}\right) \cap Es\left(\frac{1+z}{1-z}\right) = \left\{\frac{1+xz}{1-xz} : |x| = 1\right\}. \tag{3}$$

The proof that (1) holds can now be completed from (3) by appealing to the Krein - Milman theorem and the weak star compactness of the set of probability measures on $\partial\Delta$ or by apealing to Choquet's theorem [9].

Remark. The two proofs given in [1] and [4] that supp B_0 consists of all finite Blaschke products which vanish at $z = 0$ are independent of the Riesz - Herglotz representation. We note that in [6] the set supp B_0 was exactly determined by an argument that depended on the Riesz-Herglotz representation. In the previous theorem, we reversed the procedure and obtained the Riesz-Herglotz representation from the exact knowledge of supp B_0.

Finally, we give a new proof of the well known generalization of the Riesz-Helglotz reprezentation that was proved in [3].

Theorem. *If a function $f \in s\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right)$ where $\alpha \geq 1$ and $|c| \leq 1$ then is a probability measure μ on $\partial\Delta$ such that*

$$f(x) = \int_{|x|=1} \left(\frac{1+cxz}{1-xz}\right)^{\alpha} d\mu(x) \quad (|z| < 1) .$$

Proof. We assume $\alpha > 1$ since $\alpha = 1$ was essentially treated in the previous theorem. By arguing as in the proof of the previous theorem, it is clear that we need only prove that

$$\text{supp}\, s\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right) \cap EHs\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right) \subseteq \left\{\left(\frac{1+cxz}{1-xz}\right)^{\alpha} : |x| = 1\right\} .$$

To prove this inclusion suppose $f \in$ supp $s\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right)$. We have $f(x) = \left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\alpha}$ for $\phi \in$ supp B_0 [2.6]. Note that for any $|x| = 1$, $\left(\frac{1+cxz}{1-xz}\right)\left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\alpha-1}$ is in $s\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right)$ since $\alpha > 1$. Now assume that $\phi(z) \neq xz$ and write $\frac{1+c\phi(z)}{1-\phi(z)} = \sum_{k=1}^{n} \lambda_k \frac{1+cx_k z}{1-x_k z}$ where $0 < \lambda_k \leq 1$, $\sum_{k=1}^{n} \lambda_k = 1$ and $|x_k| = 1$ $(k = 1, 2, \ldots, n)$ [4]. If we write $\left(\frac{1+c\phi(z)}{1-\phi(z)}\right)^{\alpha} = \frac{1+c\phi(z)}{1-\varphi(z)}\left(\frac{1+c\phi(z)}{1-\varphi(z)}\right)^{\alpha-1}$ and use the facts mentioned above we see that $f \notin$ EHs$\left(\left(\frac{1+cz}{1-z}\right)^{\alpha}\right)$. We conclude that $\varphi(z) = xz$ and the inclusion is proved. The theorem now follows from the Krein - Milman theorem and the weak star compactness of the set of probability measures on $\partial\Delta$.

REFERENCES

[1] Abu-Muhanna, Y., *Support points of the unit ball of* $H^p (1 \leq p \leq \infty)$, Proc. Amer. Math. Soc. 89 (1983), 229–235.

[2] Brannan, D.A., Clunie, J.G., Kirwan, W.E. *On the coefficient problem for functions of bounded boundary rotation*, Ann Acad. Sci. Fenn. Ser .A.I. Math.,523 (1973).

[3] Abu–Muhanna, Y., *Variability regions and support points of subordinate families*, J. Lond. Math. Soc. (2), 29 (1984), 477 – 484.

[4] Cochrane, P.C., MacGregor, T.H., *Frechet differentiable functionals and support points for families of analytic functions*, Trans.Amer.Math.Soc.236 (1978), 75–92.

[5] Edgar, G.A., *Two integral representations*, Springer–Verlag Lecture Notes in Math. 1033 (1983), 193–198.

[6] Hallenbeck, D.J., MacGregor, T.H., *Support points of families of analytic functions described by subordination*, Trans.Amer.Math.Soc. 278 (1983), 523–546.

[7] Hallenbeck, D.J., MacGregor, T.H., *Linear problems and convexity techniques in geometric function theory*, Pitman Publishing Ltd.,(1984).

[8] Holland, F., *Extreme points of a class of functions with positive real part*, Math.Ann. 202 (1973), 85–87.

[9] Phelps, R.R., *Lectures on Choquet's Theorem*, Van Nostrand, New York, (1966).

STRESZCZENIE

W pracy podano nowy dowód reprezentacji Riesza-Herglotza. Dowód jest oparty na tym, że zbiór punktów podparcia funkcji analitycznej ograniczonej w $\Delta = \{z : |z| < 1\}$ zawiera wszystkie skończone produkty Blaschke'go.

РЕЗЮМЕ

В данной работе подано новое доказательство представления Риса–Герглотца. Доказательство опирается на том ,что множество опорных точек аналитической ограниченной функции в $\Delta = \{z : |z| < 1\}$ включает все конечные продукты Бляшке.