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An Integral Inequality for Entire Functions of Exponential Type

Nierówność całkowa dla funkcji całkowitych typu wykładniczego

Интегральное неравенство для целых функций экспоненциального типа

Govil and Jain [2, Theorem 1, inequality (1.5)] proved that if $f(z)$ is an entire function of exponential type τ, belonging to $L^\delta (1 \le \delta < \infty)$ on the real axis, $f(z) \equiv e^{i\tau z}\overline{\{f(\bar{z})\}}$, then :

$$\left(\int_{-\infty}^{\infty} |f'(x)|^\delta dx\right)^{1/\delta} \ge (1 - c_\delta^{1/\delta})\tau \left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta}, \quad (1)$$

where

$$c_\delta = 2\pi / \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^\delta d\alpha = 2^{-\delta}\sqrt{\pi}\Gamma\left(\frac{1}{2}\delta + 1\right) / \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right).$$

In this paper we observe that the inequality (1) can in fact be replaced by the sharper inequality:

$$\left(\int_{-\infty}^{\infty} |f'(x)|^\delta dx\right)^{1/\delta} \ge \left(\frac{\tau}{2}\right) \left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta}. \quad (2)$$

To show that the inequality (2) is sharper than (1) we have to show that $\left(1 - c_\delta^{1/\delta}\right) \le \frac{1}{2}$, which is equivalent to

$$\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^\delta d\alpha = 2^{\delta+1}\pi. \quad (3)$$

Since (3) is equivalent to:

$$\int_0^{2\pi} |\cos^\delta \alpha/2|\, d\alpha \leq 2\pi \,, \tag{4}$$

and (4) is evidently true, our claim that the inequality (2) is sharper then (1) is verified.

To prove (2) note that since by hypothesis $f(z) \equiv e^{i\tau z}\overline{\{f(\overline{z})\}}$, we got on differentiating with respect to z :

$$f'(z) \equiv e^{i\tau z}\overline{\{f'(\overline{z})\}} + e^{i\tau z} i\tau \overline{\{f(\overline{z})\}} \,,$$

which implies that for all real x :

$$|f'(x)| = |i\tau f(x) + f'(x)| \geq \tau |f(x)| - |f'(x)| \,,$$

which is equivalent to :

$$|f'(x)| \geq \frac{\tau}{2} |f(x)| \,, \qquad -\infty < x < \infty \,, \tag{5}$$

from which the inequality (2) follows.

Combining (2) with the inequality (1.4) of [2], we get **Theorem.** *If $f(z)$ is an entire function of exponential type τ, belonging to L^δ $(1 \leq \delta < \infty)$ on the real axis, $f(z) \equiv e^{i\tau z}\overline{\{f(\overline{z})\}}$, then for $\delta \geq 1$,*

$$\frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta} \leq \left(\int_{-\infty}^{\infty} |f'(x)|^\delta dx\right)^{1/\delta} \leq \leq \tau c_\delta^{1/\delta}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta} \,, \tag{6}$$

where

$$c_\delta = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^\delta d\alpha = 2^{-\delta}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\delta + 1\right) / \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right) .$$

If we apply the above theorem to the function $f(z) = p_n(e^{iz})$, where $p_n(z)$ is a polynomial of degree n, satisfying $p_n(z) \equiv z^n \left\{\overline{p_n(1/\overline{z})}\right\}$, we get

Corollary 1. *If $p_n(z)$ is a polynomial of degree n, satisfying $p_n(z) \equiv z^n \left\{\overline{p_n(1/\overline{z})}\right\}$, then for $\delta > 1$:*

$$\frac{n}{2}\left(\int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta\right)^{1/\delta} \leq \left(\int_0^{2\pi} |p_n'(e^{i\theta})|^\delta d\theta\right)^{1/\delta} \leq \leq n c_\delta^{1/\delta}\left(\int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta\right)^{1/\delta} \,, \tag{7}$$

where c is the same as in the above theorem.

The inequality on the right hand side of the above inequality also appears in Dewan and Govil [1].

If we make $\delta \longrightarrow \infty$ in Corollary 1, we get

Corollary 2. *If $p_n(z)$ is a polynomial of degree n, satisfying $p_n(z) \equiv z^n \left\{\overline{p_n(1/\bar{z})}\right\}$, then*

$$\max_{|z|=1} |p_n'(z)| = \frac{n}{2} \max_{|z|=1} |p_n(z)|. \tag{8}$$

The above corollary was proved independently by O'Hara and Rodriguez [3, Theorem 1] and by Saff and Shiel-Small [4, Theorem 7].

REFERENCES

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STRESZCZENIE

Funkcja całkowita $f(z)$ typu wykładniczego τ należąca do $L^\delta(1 \le \delta < \infty)$ na osi rzeczywistej $f(z) \equiv e^{i\tau z}\overline{\{f(\bar{z})\}}$ dla $\delta \ge 1$ spełnia nierówność

$$\frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta} \le \left(\int_{-\infty}^{\infty} |f'(x)|^\delta dx\right)^{1/\delta} \le \tau c_\delta^{1/\delta}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta},$$

gdzie

$$c_\delta = 2\pi / \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^\delta d\alpha = 2^{-\delta}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\delta + 1\right) / \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right).$$

Nierówność ta poprawia znane nierówności tego typu i zawiera jako specjalne przypadki pewne znane wyniki tego typu.

РЕЗЮМЕ

Целая функция экспоненциального типа τ класса $L^\delta(1 \le \delta < \infty)$ на вещественной оси, $f(z) \equiv e^{i\tau z}\overline{\{f(\bar{z})\}}$ для $\delta \ge 1$ исполняет неравенство

$$\frac{\tau}{2}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta} \le \left(\int_{-\infty}^{\infty} |f'(x)|^\delta dx\right)^{1/\delta} \le \tau c_\delta^{1/\delta}\left(\int_{-\infty}^{\infty} |f(x)|^\delta dx\right)^{1/\delta},$$

где

$$c_\delta = 2\pi / \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^\delta d\alpha = 2^{-\delta}\sqrt{\pi}\Gamma\left(\frac{1}{2}\delta + 1\right) / \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right) .$$

Это неравенство улучшает известные неравенства этого типа и включает как частные случаи некоторые известные результаты.