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Geometric Interpretation of Curvatures in the 2-dimensional Special Kawaguchi Spaces

Geometryczna interpretacja krzywizny w specjalnych 2-wymiarowych przestrzeniach Kawaguchiego

Геометрическая интерпретация кривизны в специальных 2-мерных пространствах Кавагухи

Introduction. We will consider the following bilinear and quadratic forms in R^2:

$$< x,y >= x^1y^1 + x^2y^2 ,$$

$$(x,y) = x^1y^2 - x^2y^1 ,$$

$$p(x) =< x, Px >$$

where P is a fixed symmetric and nonsingular matrix.

By G_p we denote the subgroup of GL_2 defined as follows:

$$G_p = \{A \in GL_2 : p(AX) = (detA)p(x) , \text{ for } x \in R^2\} . \tag{1}$$

In this paper we will consider the group of affine transformations of R^2:

$$x \mapsto Ax + a , \qquad A \in G_p \tag{2}$$

and the plane curves with the arc length defined by the formula:

$$ds = \begin{cases} \frac{(x,\dot{x})}{p(x)}dt & \text{if } p(x) \neq 0 \\ 0 & \text{if } p(x) = 0 . \end{cases} \tag{3}$$

In the centroaffine case and:

$$ds = \begin{cases} \frac{(\dot{x},\ddot{x})}{p(\dot{x})}dt & \text{if } p(\dot{x}) \neq 0 \\ 0 & \text{if } p(\dot{x}) = 0 \,. \end{cases} \tag{4}$$

in the general case.

The pair (R^2, ds) is the 2-dimensional special Kawaguchi space [1], [2].

In this paper we shall give the geometric interpretation of the curvature of a plane curve. Moreover, we shall find Frenet's formulas [1], [2] and curves with a constant curvature.

2. The centroaffine curvature. We note that for arbitrary $x, y \in R^2$ we have:

$$< x, y > = (x, Jy)\,, \tag{5}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In this paragraph we will consider centroaffine transformations and curves $t \mapsto x(t)$ such that $p(x) \neq 0$.

By s we denote the natural parameter of x. Then we have:

$$\frac{(x, x')}{< x, Px >} = 1\,.$$

Making use of (5) in the above condition we can rewrite it in the form:

$$(x, x' - JPx) = 0\,.$$

Hence

$$x' = \kappa x + JPx\,. \tag{6}$$

We will call the function κ a centroaffine curvature of a curve x.

Lemma 1. *The centroaffine curvature κ of a curve $x : s \mapsto x(s)$ is given by the formula:*

$$\kappa(s) = \frac{(p \circ x)'(s)}{2(p \circ x)(s)}\,. \tag{7}$$

Proof. Let $\Delta = detP$. It is easy to verify that:

$$JPJPx = -\Delta x. \tag{8}$$

The conditions (5) and (6) imply:

$$\kappa = \frac{(x', JPx)}{p(x)}\,. \tag{9}$$

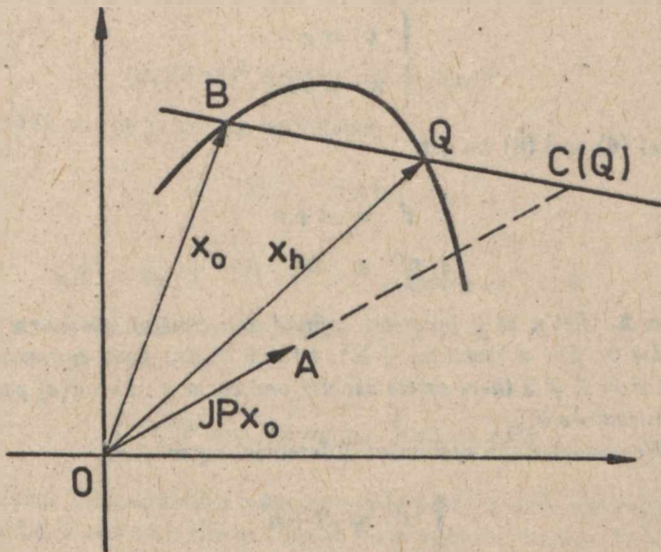
Making use of (6), (8) and (9) we obtain:

$$\begin{aligned} (p \circ x)' &= (x, JPx)' = (x', JPx) + (x, JPx') = \\ &= (x', JPx) + (x, \kappa JPx + JPJx) = \\ &= (x', JPx) + \kappa(x, JPx)\,. \end{aligned}$$

Hence due to (6) we get (7).

Now we give the geometric interpretation of the centroaffine curvature κ in an arbitrary parametrisation.

Let $x_0 = x(t_0)$, $x_h = x(t_0 + h)$. We denote by A, B, Q the ends of the vectors JPx_0, x_0, x_h, respectively. Further, let $C(Q)$ denote the points of intersection of straight lines BQ and OA.



We prove that:

$$\kappa(t_0) = \lim_{P \to B} \frac{{}^*\text{area } \Delta AOB}{{}^*\text{area } \Delta BOC(P)}, \tag{10}$$

where ${}^*\text{area } \Delta PQR = \frac{1}{2}(\overrightarrow{QP}, \overrightarrow{QR})$.

The curvature κ in an arbitrary parametrisation is given by the formula:

$$\kappa(t) = \frac{(\dot{x}, JPx)}{(x, \dot{x})}. \tag{11}$$

We have:

$$x_0 + \lambda(x_h - x_0) = \mu JPx_0$$

for some λ and μ. Hence:

$$\mu = \frac{(x_h - x_0, x_0)}{(x_h - x_0, JPx_0)}.$$

Using Taylor's expansion we get:

$$\frac{{}^*\text{area } \Delta AOB}{{}^*\text{area } \Delta BOC(P)} = \frac{(JPx_0, x_0)}{(x_0, \mu JPx_0)} =$$

$$= \frac{(JPx_0, x_0)}{(x_0, JPx_0)} \frac{(\dot{x}_0, JPx_0)h + \cdots}{(\dot{x}_0, x_0)h + \cdots} \longrightarrow \frac{(\dot{x}_0, JPx_0)}{(x_0, \dot{x}_0)} = \kappa(t_0).$$

We denote by Z the point of intersection (if it exists) of the tangent to z at the point B and the straight line OA. It is easy to see that:

$$\overrightarrow{OZ} = -\frac{1}{\kappa} JPz_0 . \tag{12}$$

3. The counterpart of Frenet's formulas. Let

$$\begin{cases} t = z \\ n = JPz . \end{cases} \tag{13}$$

Making use of (6) and (8) we get:

$$\begin{cases} t' = \kappa t + n \\ n' = -\Delta t + \kappa n . \end{cases} \tag{14}$$

Theorem 2. *Let κ be a function defined and continuous in an open interval which contains 0. For a given $z_0 \in R^2$, $t_0 \neq 0$ and a fixed symmetric matrix P such that $\det P = \Delta \neq 0$ there exists exactly one curve $z : s \mapsto z(s)$ passing through z_0 with the curvature κ.*

Proof. We consider the system of differential equations:

$$\begin{cases} t' = \kappa t + n \\ n' = -\Delta t + \kappa n \end{cases}$$

with the initial condition $n_0 = JPt_0$, $(t_0, n_0) \neq 0$.

Making use of (8) and (14) we obtain:

$$\begin{aligned} (n - JPt)' &= -\Delta t + \kappa n - \kappa JPt - JPn = \\ &= (JP - \kappa I)JPt + (\kappa I - JP)n = \\ &= (\kappa I - JP)(n - JPt) . \end{aligned}$$

The above differential equation and the initial condition imply $n = JPt$. Moreover $(t, n) \neq 0$ follows from the differential equation $(t,n)' = 2\kappa(t,n)$ and the initial condition.

The curve:

$$z(s) = t(s) - n(0) + z_0 \tag{15}$$

has required properties.

4. Curves with constant centroaffine curvature. The solution of the equation (6) $z' = \kappa z + JPz$, ($\kappa(s) = k = const$) which passes through a point $z_0 \in R^2$, $p(z_0) \neq 0$ is of the form:

$$z(s) = e^{ks} \exp(sJP) z_0 . \tag{16}$$

We find curves with $\kappa \equiv 0$. Let $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $x_0 = \begin{pmatrix} p \\ q \end{pmatrix}$.

1° $\Delta = \det P > 0$. Let $\delta = \sqrt{\Delta}$. Due to (8) we have:

$$\exp(sJP)x = \cos\delta s\, x + \frac{1}{\delta}\sin\delta s\, JPx.$$

Hence we have:

$$(x, JPx_0)^2 + \Delta(x_0, x)^2 = p(x_0)^2\,. \tag{17}$$

The equation (17) can be rewritten as follows:

$$p(x) = p(x_0) \tag{18}$$

or

$$aX^2 + 2bXY + cY^2 - (ap^2 + 2bpq + cq^2) = 0 \tag{19}$$

This equation represents an ellipse with the center at 0.

2° $\Delta < 0$.Let $\sigma = \sqrt{-\Delta}$. We have:

$$\exp(sJP)x = \text{ ch } \sigma s\, x + \frac{1}{\delta}\text{ sh } \sigma s\, JPx$$

Hence we get (18). This equation represents a hyperbola with the center at O.

Example. Let's consider the quadratic form $p(x) = < x, x >$. It is easy to see that:

$$G_p = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 > 0 \right\}.$$

The arc length of a curve $t \mapsto x(t)$ is given by the formula

$$ds = \frac{(x, \dot{x})}{< x, x >}dt\,.$$

The circles with the center at O are curves with the centroaffine curvature $\kappa \equiv 0$. The equation (19) has the form:

$$X^2 + Y^2 = p^2 + q^2.$$

We note that the vector JPx_0 is parallel to the tangent at x_0.

5. The general case. In the general case for the natural parameter s we have:

$$\frac{(x', x'')}{p(x')} = 1$$

Hence

$$x'' = \lambda x' + JPx'\,; \tag{20}$$

the function λ will be called a curvature. Consider the indicatrix of tangents of the curve x [3]. We denote by $\hat{s}$ and κ the centroaffine arc length and centroaffine curvature of the indicatrix respectively.Using (20) we obtain:

$$\frac{d\hat{s}}{ds} = \frac{\left(x', \frac{d}{ds}x'\right)}{(x', JPx')} = \frac{(x', \lambda x' + JPx')}{(x', JPx')} = 1$$

Thus $d\hat{s} = ds$. Moreover, we have:

$$\kappa = \frac{\left(\frac{d}{ds}x', JPx'\right)}{(x', JPx')} = \frac{(\lambda x' + JPx', JPx')}{(x', JPx')} = \lambda$$

It means that the curvature of a curve coincides with the centroaffine curvature of its indicatrix.

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STRESZCZENIE

W pracy tej podajemy geometryczną interpretację krzywizny krzywych płaskich w specjalnych 2-wymiarowych przestrzeniach Kawaguchiego. Ponadto podajemy reper Freneta i znajdujemy krzywe o stałej krzywiźnie.

РЕЗЮМЕ

В данной работе представлена геометрическая интерпретация кривизны плоских кривых в специальных 2-мерных пространствах Кавагухи. Найдено также репер Френета и кривые с постоянной кривизной .