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Another Proof of Kneser's Theorem for Generalized Differential Equation

Inny dowód twierdzenia Knesera dla uogólnionego równania różniczkowego

It is well known that Kneser's theorem for the differential equation $x' = f(t, x)$, where $(t, x) \in R \times R^n$, is equally valid for the so-called generalized equations, i.e. paratingent equation $(Px)(t) \subset F(t, x)$, contingent equation $(Cx)(t) \subset F(t, x)$ and differential inclusion $x' \in F(t, x)$. But in the case of generalized equations the proofs of the theorem (cf. [3], [4], [8], [10], [11], [15]) are by no means so clear as for ordinary differential equations. In the present paper we shall show that Kneser's theorem for generalized differential equations may by proved by Miller's method (cf. [7]), losing nothing of its clarity.

1. Preliminaries. Let R be a real line and R^n be the euclidean n-dimensional space with usual norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, where $x = (x_1, \dots, x_n)$. The family of all nonempty compact nad convex subsets of R^n is denoted by Conv R^n. $K_X(a; r)$ is the ball with its center at point $x \in X$ and a radius r in a given metric space X, and $K_X(A; r) = \bigcup_{a \in A} K_X(a; r)$ for $\emptyset \neq A \subset X$.

Let $I = [0, 1] \subset R$ be the unit compact interval, and C_I be the Banach space of all continuous functions $\varphi : I \to R^n$ with supremum norm $\|\cdot\|$.

If $\varphi \in C_I$, then for $t_0 \in I$ the „paratingent" or „paratingent derivative" (respectively „contingent" or „contingent derivative") of φ at t_0 is defined as the set of all points $x \in R^n$ for which there exist two sequences of values $t_i \in I$, $s_i \in I$, where $t_i \neq s_i$, both sequences convergent to t_0 and such that $x = \lim_{i \to \infty} \dfrac{\varphi(t_i) - \varphi(s_i)}{t_i - s_i}$ (respectively for „contingent", there exists a sequence of values $t_i \in I$ distinct from t_0 convergent to t_0 and such that $x = \lim_{i \to \infty} \dfrac{\varphi(t_i) - \varphi(t_0)}{t_i - t_0}$); the paratingent (contingent) derivative of φ at t is denoted by $(P\varphi)(t)$ $((C\varphi)(t))$. Having a multifunction $F: I \times R^n \to \operatorname{Conv} R^n$ we understand by the paratingent equation (respectively – the contingent equation and the differential inclusion) a relation

$$(Px)(t) \subset F(t, x(t))\ ((Cx)(t) \subset F(t, x(t)), x'(t) \in F(t, x(t))).$$

By a solution of this paratingent (contingent) equation we understand a function $\varphi \in C_I$ whose paratingent (contingent) at each point $t \in I$ lies in the given set $F(t, \varphi(t))$ while a solution of a differential inclusion is an absolutely continuous function $\varphi \in C_I$ for which $\varphi'(t) \in F(t, \varphi(t))$ almost everywhere on I in the sense of Lebesque measure.

The multifunction $F: I \times R^n \to \operatorname{Conv} R^n$ is called upper semi-continuous (abbreviated as usc) if for every $(t, x) \in I \times R^n$ and for every $\epsilon > 0$ there exists $\delta > 0$, such that $F(s, y) \subset K_{R^n}(F(t, x); \epsilon)$ for each $(s, y) \in K_{R^{1+n}}((t, x); \delta)$; if additionally the inclusion $F(t, x) \subset K_{R^n}(F(s, y); \epsilon)$ is satisfied for each $(s, y) \in K_{R^{1+n}}((t, x); \delta)$, then the multifunction F is continuous. As Ważewski pointed out in [13] and [14], under the assumed usc of F, the paratingent and contingent equations are equivalent to the differential inclusion, i.e. a continuous function φ satisfies $(P\varphi)(t) \subset F(t, \varphi(t))$ or $(C\varphi)(t) \subset F(t, \varphi(t))$ if and only if it is absolutely continuous and $\varphi'(t) \in F(t, \varphi(t))$ a.e. on I. Therefore every theorem concerning the properties of solutions of the paratingent (contingent) equation is at the same time a theorem on the properties of solutions of the differential inclusion and vice versa.

Throughout this paper we shall assume that the multifunction $F: I \times R^n \to \operatorname{Conv} R^n$ is usc and satisfies the following condition: $F(t, x) \subset K_{R^n}(\theta, m(t))$, $(t, x) \in I \times R^n$, where $\theta = (0, \dots, 0)$ is an origin of R^n and $m: I \to [0, \infty)$ is a fixed continuous function.

The set of all the solutions of the initial value problem

(1) $$x'(t) \in F(t, x(t)), t \in I,$$

(2) $$x(0) = x_0, x_0 \in R^n,$$

will be denoted by $\mathcal{E}(F, x_0)$ (this set $\mathcal{E}(F, x_0)$ is called the emission of the initial point x_0 on account of equation (1) by some authors (see [3], [9])).

Finally, let us introduce still one more designation

$$B = \overline{K_{R^n}(x_0, r_0)}, \text{ where } r_0 = |x_0| + 3\int_0^1 (m(t) + 1)\,dt$$

and $\bar{K}$ denoted the closure of K.

2. Some facts from the theory of ordinary differential and paratingent equations. Below there are three theorems which will be useful in the last section of this paper.

Theorem 1 ([9, Théorème III]). *$\mathcal{E}(F, x_0)$ is a nonempty compact subset of C_I.*

Theorem 2 ([10, Lemme 2]). *There exists a sequence of continuous multifunctions $F_i: I \times R^n \to \operatorname{Conv} R^n$, $i = 1, 2, \dots$, such that*

1° $F_{i+1}(t, x) \subset F_i(t, x) \subset K_{R^n}(\theta, m(t)+1), (t, x) \in I \times R^n$,

2° $F(t, x) \subset F_i(t, x)$ *for* $(t, x) \in I \times B$,

3° $F(t, x) = \bigcap_{i=1}^{\infty} F_i(t, x)$ *for* $(t, x) \in I \times B$.

Theorem 3 ([9, Théorème VI]). *If multifunctions F_i are the same as in Theorem 2, then*

$$\mathcal{E}(F_{i+1}, x_0) \subset \mathcal{E}(F_i, x_0)$$

and

$$\mathcal{E}(F, x_0) = \bigcap_{i=1}^{\infty} \mathcal{E}(F_i, x_0).$$

Now we shall recall some facts from the theory of ordinary differential equations. Because at present they are sufficiently well-known we omit their detailed proofs. Thus, let us suppose that there is a given function $f: I \times R^n \to R^n$ which is Lebesque measurable in t for each $x \in R^n$ and continuous in x for each $t \in I$. This function is called a function of Caratheodory's type. Let us assume that f satisfies the inequality $|f(t, x)| \leqslant m(t)$, $(t, x) \in I \times R^n$. Then the initial value problem (abbreviated as ivp)

(3f) $$x'(t) = f(t, x(t)), t \in I,$$

(2) $$x(0) = x_0$$

has at least one solution defined on the whole interval I (by the solution of ivp (3f) (2) we mean every function $\varphi \in C_I$ such that is absolutely continuous and satisfies equation (3f) a.e. in I). This solution is bounded and lipschitzean and more precisely if $\varphi \in C_I$ is the solution of ivp (3f) (2), then

(a) $$\|\varphi\| \leqslant |x_0| + \int_0^1 m(t)\, dt$$

(b) $$|\varphi(t) - \varphi(s)| \leqslant \max_{\tau \in I} m(\tau)\, |t - s|,\ t, s \in I.$$

3. Approximation theorems. For the convenience of reader first we shall recall two theorems in the form sufficient for our considerations.

Theorem 4 (Lasota – Yorke [11]). *If $f: I \times R^n$ is continuous, then for every $\epsilon > 0$ there exists a locally lipschitzean function $f_\epsilon: I \times R^n \to R^n$ such that*

$$\sup_{(t,x)\in I\times R^n} |f(t,x)-f_\epsilon(t,x)|<\epsilon.$$

Theorem 5 (Alexiewicz – Orlicz [1]). *If $f: I\times B\to R^n$ of Caratheodory's type satisfying the condition $|f(t,x)|\leqslant m(t)$ for $(t,x)\in I\times B$, then there exists a sequence of continuous functions $f_i: I\times B\to R^n$ such that $|f_i(t,x)|\leqslant m(t)$, $(t,x)\in I\times B$, $i=1,2,\ldots$ and*

$$\lim_{i\to\infty}\sup_{x\in B}|f_i(t,x)-f(t,x)|=0 \text{ for almost all } t\in I.$$

Theorem 6. *Let $f: I\times R^n\to R^n$ be a Caratheodory's type function satisfying the condition $|f(t,x)|\leqslant m(t)$ for $(t,x)\in I\times R^n$ and let $\varphi\in C_I$ be a solution of ivp (3_f) (2). Then there exists a sequence of Caratheodory's type functions $f_i: I\times R^n\to R^n$ satisfying the condition $|f_i(t,x)|\leqslant 3(m(t)+1)$ such that function φ is the unique solution of ivp (3_{f_i}) (2), where*

(3_{f_i}) $$x'(t)=f_i(t,x(t)),\ t\in I,\ i=1,2,\ldots;$$

moreover

(4) $$\lim_{i\to\infty}\sup_{x\in B}|f_i(t,x)-f(t,x)|=0 \text{ for almost all } t\in I.$$

Proof. In view of Theorem 5 there exists a sequence of continuous functions $g_i: I\times B\to R^n$ such that

(*) $$\lim_{i\to\infty}\sup_{x\in B}|f(t,x)-g_i(t,x)|=0 \text{ for almost } t\in I$$

and

(**) $$|g_i(t,x)|\leqslant m(t),\ (t,x)\in I\times B,\ i=1,2,\ldots.$$

Functions $g_i^*: I\times R^n\to R^n$ defined by formula

$$g_i^*(t,x)=\begin{cases} g_i(t,x), \text{ if } |x|\leqslant r_0=|x_0|+3\int_0^1(m(t)+1)\,dt,\\ g_i(t,r_0x/|x|), \text{ if } |x|>r_0\end{cases}$$

are a continuous extension of g_i to $I\times R^n$ and still satisfying the inequality (**).

From Theorem 4 it further follows that for each function g_i there exists a locally lipschitzean function $h_i: I\times R^n\to R^n$ such that

$$\sup_{(t,x)\in I\times R^n} |g_i^*(t,x)-h_i(t,x)| < (1/2)i,\ i=1,2,\dots.$$

The restriction of each h_i to $I \times B$, i.e. the function $h_{i|I\times B}$, satisfies the global Lipschitz condition with some constant L_i. Now we extend every restriction $h_{i|I\times B}$, using the same technique as before, to a function $h_i^* : I \times R^n \to R^n$ and then define the function $f_i : I \times \times R^n \to R^n$ by formula

$$f_i(t,x) = h_i^*(t,x) - h_i^*(t,\varphi(t)) + f(t,\varphi(t)),\ (t,x)\in I\times R^n,\ i=1,2,\dots.$$

Measurability of $f(\cdot,x)$ is obvious. We have

$$|f_i(t,x)| \leqslant |h_i^*(t,x)| + |h_i^*(t,\varphi(t))| + |f(t,\varphi(t))| \leqslant 3(m(t)+1).$$

On the other hand, h_i^* satisfies the global Lipschitz condition with constant L_i with respect to second variable because $|x-y| \geqslant r_0\, |x/|x| - y/|y||$ for $x,y\notin B$ and h_i is lipschitzean with L_i constant. Thus $|f_i(t,x)-f_i(t,y)| \leqslant |h_i^*(t,x)-h_i^*(t,y)| \leqslant \leqslant L_i|x-y|$ for $(t,x),(t,y)\in I\times R^n$ and therefore every ivp (3_{f_i}) (2) has exactly one solution. But for almost each $t\in I$

$$f_i(t,\varphi(t)) = h_i^*(t,\varphi(t)) - h_i^*(t,\varphi(t)) + f(t,\varphi(t)) = \varphi'(t)$$

hence φ is this unique solution.

There still remains to prove (4). We have

$$0 \leqslant \sup_{x\in B} |f_i(t,x)-f(t,x)| \leqslant \sup_{x\in B} (|h_i^*(t,x)-f(t,x)| +$$

$$+ |h_i^*(t,\varphi(t)) - f(t,\varphi(t))|) \leqslant 2\sup_{x\in B} |g_i(t,x)-f(t,x)| + 1/i,\ i=1,2,\dots$$

hence in view of condition (*) $\lim_{i\to\infty}\sup_{x\in B} |f_i(t,x)-f(t,x)| = 0$ for almost every $t\in I$ which completes the proof of the theorem.

Theorem 7. *Let $f: I\times R^n \to R^n$ and $f_i : I\times R^n\to R^n$, $i=1,2,\dots$, be Caratheodory's type functions satisfying conditions $|f(t,x)|\leqslant m(t)$, $|f_i(t,x)|\leqslant 3(m(t)+1)$, $(t,x)\in \in I\times R^n$, and such that*

$$(5)\qquad \lim_{i\to\infty}\sup_{x\in B} |f(t,x)-f_i(t,x)| = 0 \textit{ for almost every } t\in I.$$

Let $\varphi_i\in C_I$, $i=1,2,\dots$, be the solution of ivp (3_{f_i}) (2).

Then there exists a subsequence $\{\varphi_{i_j}\}$ uniformly convergent to a function $\varphi\in C_I$ which is the solution of ivp (3_f) (2).

If additionally the problem (3f) (2) has a unique solution, then the whole sequence $\{\varphi_i\}$ uniformly converges to φ.

Proof. The functions φ_i are uniformly bounded and uniformly continuous because

$$|\varphi_i(t)| \leqslant |x_0| + 3\int_0^t (m(\tau)+1)\,d\tau$$

and
$$|\varphi_i(t) - \varphi_i(s)| \leqslant \max_{\tau\in I}(3\,m(\tau)+3)\,|t-s|$$

Thus there exists a subsequence $\{\varphi_{ij}\}$ uniformly convergent to some function $\varphi \in C_I$. We will show that φ is the solution of ivp (3f) (2). We have

$$\varphi(t) - [x_0 + \int_0^t f(s,\varphi(s))\,ds] = \varphi(t) - [x_0 + \int_0^t \{f(s,\varphi(s)) -$$

$$-f(s,\varphi_{i_j}(s)) + f(s,\varphi_{i_j}(s)) + f_{i_j}(s,\varphi_{i_j}(s)) - f_{i_j}(s,\varphi_{i_j}(s))\}\,ds] =$$

$$= \underbrace{\varphi(t) - (x_0 + \int_0^t f_{i_j}(s,\varphi_{i_j}(s))\,ds}_{\alpha_j(t)} + \int_0^t [f(s,\varphi(s)) - f(s,\varphi_{i_j}(s))]\,ds +$$

$$+ \underbrace{\int_0^t [f_{i_j}(s,\varphi_{i_j}(s)) - f(s,\varphi_{i_j}(s))]\,ds)}_{-\gamma_j(t)} = \alpha_j(t) + \beta_j(t) + \gamma_j(t),\ t\in I.$$

Since $\alpha_j(t) = \varphi(t) - \varphi_{ij}(t)$, then $\alpha_j(t) \to 0$ as $j\to\infty$. Similarly, in view of the continuity of f with respect to second variable and the Lebesque's Dominated cobvergence Theorem, the value $\beta_j(t)$ converges to zero when $j\to\infty$.

We also assert that $\gamma_j(t)$ converges to 0 as $j\to\infty$. Indeed, in virtue of the limit condition (5) we have

$$0 \leqslant |\gamma_j(t)| \leqslant \int_0^t |f_{i_j}(s,\varphi_{i_j}(s)) - f(s,\varphi_{i_j}(s))|\,ds \leqslant$$

$$\leqslant \int_0^t \sup_{x\in B} |f_j(s,x) - f(s,x)|\,ds \to 0, \text{ as } j\to\infty.$$

Therefore it must be

$$\varphi(t) = x_0 + \int_0^t f(s,\varphi(s))\,ds,\ t\in I,$$

which means that φ is the solution of ivp (3f) (2). If we assume now that ivp (3f) (2) has exactly one solution, then every subsequence $\{\varphi_{i_j}\}$ contains a subsequence $\{\varphi_{i_j}\}$ converging to this unique solution. Thus the whole sequence $\{\varphi_i\}$ converges to this solution. The proof of the theorem is completed.

5. The generalized Kneser's theorem. A function $f : I \times R^n \to R^n$ is called the selector of multifunction $F : I \times R^n \to \operatorname{Conv} R^n$ if $f(t, x) \in F(t, x)$ for $(t, x) \in I \times R^n$.

Lemma 1. *Let multifunction $F : I \times R^n \to \operatorname{Conv} R^n$ be continuous and satisfy the condition $F(t, x) \subset K_{R^n}(\theta, m(t))$, and let $\varphi \in C_I$ be a solution of* ivp (1) (2). *Then there exists a Caratheodory's type selector f of F such that φ is the solution of* ivp (3_f) (2). *Moreover $|f(t, x)| \leqslant m(t)$ for $(t, x) \in I \times R^n$.*

Proof. For $(t, x) \in I \times R^n$ let us define

$$f(t, x) = \begin{cases} \operatorname{proj}(\varphi'(t) \mid F(t, x)) & \text{when } \varphi'(t) \text{ exists.} \\ \operatorname{proj}(\theta \mid F(t, x)) & \text{when } \varphi'(t) \text{ does not exist,} \end{cases}$$

where $\operatorname{proj}(z \mid A)$ denotes the metric projection of a point $z \in R^n$ onto a nonempty compact convex subset A of R^n (in case of euclidean norm in R^n this projection is always a one-point set). Therefore f is a selector of F and obviously satisfies the inequality $|f(t, x)| \leqslant m(t)$. By Berge's theorem [2, Th 3, Chapter VI] $f(t, \cdot)$ is continuous for every $t \in I$ and by Castaing's theorem [5, Th 5.1] $f(\cdot, x)$ is measurable for every $x \in R^n$. Thus f is Caratheodory's type function. Moreover, for almost every $t \in I$ $\varphi'(t) \in F(t, \varphi(t))$, hence $\varphi'(t) = f(t, \varphi(t))$ which completes the proof of the theorem.

Theorem 8. *If multifunction $F : I \times R^n \to \operatorname{Conv} R^n$ is continuous and $F(t, x) \subset K_{R^n}(\theta, m(t))$, then the set $\mathcal{E}(F, x_0)$ is a continuum.*

Proof. We must prove only the connectedness of $\mathcal{E}(F, x_0)$ because by Theorem 1 it is nonempty and compact. Let us suppose the contrary, i.e. that $\mathcal{E}(F, x_0)$ is not connected.. Then $\mathcal{E}(F, x_0) = E_0 \cup E_1$ where E_0, E_1 are nonempty, disjoint closed subsets of C_I. Then od $(E_0, E_1) = \inf \{\| u - v \| : u \in E_0, v \in E_1\} = d > 0$. Let us define the function $k : C_I \to R$ by formula

$$k(u) = d(u, E_0) - d(u, E_1)$$

where $d(u, E) = \inf \{\| u - v \| : v \in E\}$.
Moreover

$$k(u) = \begin{cases} -d(u, E_1) \leqslant -d, & \text{if } u \in E_0, \\ d(u, E_0) \geqslant d, & \text{if } u \in E_1. \end{cases}$$

Thus if $u \in \mathcal{E}(F, x_0)$ then still $k(u) \neq 0$.

Let u^0 and u^1 be two solutions of ivp (1) (2) such that $u^0 \in E_0$ and $u^1 \in E_1$.

By Lemma 1 there exist selectors f^0, f^1 of multifunction F such that u^0 and u^1 is the solution of ivp (3_{f^0}) (2) and (3_{f^1}) (2) respectively. f^0 and f^1 are the functions of Caratheodory's type and satisfy the condition $|f^j(t, x)| \leqslant m(t), j = 0, 1, (t, x) \in I \times R^n$. Thus, in view of Theorem 6 there exist sequences $\{f_i^j\}$, $i = 1, 2, \ldots, j = 0, 1$ of functions $f_i^j : I \times R^n \to R^n$ such that

1) f_i^j is Caratheodory's type and satisfies the inequality $|f_i^j(t, x)| \leqslant 3(m(t) + 1), i = 1, 2, \ldots, j = 0, 1,$

2) u_i^j is unique the solution of ivp $(3_{f_i^j})$ (2), $i = 1, 2, \ldots, j = 0, 1,$

3) $\lim_{i \to \infty} \sup_{x \in B} |f_i^j(t, x) - f(t, x)| = 0$ for almost every $t \in I, j = 0, 1$.

For $i = 1, 2, \ldots$ and $\alpha \in I$ let us put

$$f_i^\alpha(t, x) = (1 - \alpha) f_i^0(t, x) + \alpha f_i^1(t, x), (t, x) \in I \times R^n$$

and consider such a defined family of functions f_i^α. First of all we conclude that for arbitrarily fixed $\alpha, \beta \in I$

$$|f_i^\alpha(t, x) - f_i^\beta(t, x)| \leqslant |\beta - \alpha| \, |f_i^0(t, x) - f_i^1(t, x)| \leqslant 6 |\beta - \alpha| (m(t) + 1).$$

Hence

(6) $$\sup_{x \in B} |f_i^\alpha(t, x) - f_i^\beta(t, x)| \leqslant 6 |\beta - \alpha| (m(t) + 1).$$

In virtue of 3) we have

(7) $$\lim_{i \to \infty} \sup_{x \in B} |f_i^\alpha(t, x) - f^\alpha(t, x)| = 0 \text{ for almost every } t \in I$$

where $f^\alpha = (1 - \alpha) f^0 + \alpha f^1$.

Moreover, every f_i^α satisfies the global Lipschitz condition with respect to x and with some constant L_i^α which is no large then $L_i = \max(L_i^0, L_i^1)$. Therefore, there exists exactly one solution u_i^α of ivp $(3_{f_i^\alpha})$ (2). We assert that for every fixed i the solution u_i^α continuously depends on the parameter α.

Indeed, we have

$$\begin{aligned} |u_i^\alpha(t) - u_i^\beta(t)| &\leqslant \int_0^t |f_i^\alpha(s, u_i^\alpha(s)) - f_i^\beta(s, u_i^\beta(s))| \, ds \leqslant \\ &\leqslant \int_0^t |f_i^\alpha(s, u_i^\alpha(s)) - f_i^\alpha(s, u_i^\beta(s))| \, ds + \\ &+ \int_0^t |f_i^\alpha(s, u_i^\beta(s)) - f_i^\beta(s, u_i^\beta(s))| \, ds \leqslant \\ &\leqslant L_i^\alpha \int_0^t |u_i^\alpha(s) - u_i^\beta(s)| \, ds + 6 |\beta - \alpha| \int_0^t (m(s) + 1) \, ds \leqslant \end{aligned}$$

$$\leqslant 6\,|\beta-\alpha|\int_0^1 (m(s)+1)\,ds + L_i\int_0^t |u_i^\alpha(s)-u_i^\beta(s)|\,ds,\ t\in I,$$

and by Gronwall's Lemma ([7])

$$\|u_i^\alpha - u_i^\beta\| \leqslant 6\,|\beta-\alpha|\,c\,e^{L_i t},\ t\in I.$$

Thus if $\beta\to\alpha$ then u_i^β uniformly converges to u_i^α. Then it follows that for $i=1,2,\dots$, $k(u_i^\alpha)$ is the continuous function of α. Since $u_i^0=u^0$, $u_i^1=u^1$, $k(u^0)<$ and $k(u^1)>0$. The sequence $\{\alpha_i\}$ is bounded and therefore it contains a subsequence $\{\alpha_{i_j}\}$ which is convergent to $\bar\alpha$. Let us choose an arbitrary $\epsilon>0$. In view of (6) and (7), for almost every $t\in I$ and for sufficiently large j we have

$$\sup_{x\in B}|f_{i_j}^{\alpha_{i_j}}(t,x)-f^{\bar\alpha}(t,x)| \leqslant \sup_{x\in B}|f_{i_j}^{\alpha_{i_j}}(t,x)-f_{i_j}^{\bar\alpha}(t,x)| +$$

$$+\sup_{x\in B}|f_{i_j}^{\bar\alpha}(t,x)-f^{\bar\alpha}(t,x)|<\epsilon.$$

Thus it must be that

$$\lim_{j\to\infty}\sup_{x\in B}|f_{i_j}^{\alpha_{i_j}}(t,x)-f^{\bar\alpha}(t,x)|=0,\ \text{a.e. on } I.$$

Thus there exists a subsequence $\{m\}$ of sequence $\{i_j\}$ such that the solution $u_m^{\alpha m}$ of ivp $(3_{f_m^{\alpha m}})$ (2) uniformly converges to a solution $\bar u$ of ivp $(3_{f\bar\alpha})$ (2). Since $f^\alpha(t,x)=(1-\bar\alpha)f^0(t,x)+\bar\alpha f^1(t,x)\in F(t,x)$ for $(t,x)\in I\times R^n$ then $\bar u$ is the solution of ivp (1) (2) which means that $\bar u\in \mathcal{E}(F,x_0)$.

Thus it must be that $k(\bar u)\neq 0$. But for this sequence $\{u_m^{\alpha m}\}$ of solutions it is always $k(u_m^{\alpha m})=0$, $m=1,2,\dots$, and hence $\lim_{m\to\infty} k(u_m^{\alpha m})=k(\bar u)=0$. This contradiction proves that $\mathcal{E}(F,x_0)$ is a continuum and the proof of our theorem is completed.

From the above Theorem 8, Theorems 2 and 3 and from the fact if an intersection $\bigcap_{i=1}^\infty C_i$ of the decreasing sequence of continuum C_i is a continuum (cf. [6, Corollary 2, p. 430]) the generalized Kneser's theorem follows immediately:

If multifunction $F: I\times R^n\to \operatorname{Conv} R^n$ is usc and satisfies the condition $F(t,x)\subset \subset K_{R^n}(\theta, m(t))$, $(t,x)\in I\times R^n$, then the emission $\mathcal{E}(F,x_0)$ is a continuum in Banach space C_I.

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STRESZCZENIE

Udowodniono, że zbiór rozwiązań równiania $x' \in F(t, x)$ spełniających warunek początkowy $x(0) = x_0$, gdzie F jest multifunkcją górnie półciągłą o wartościach zwartych i wypukłych, jest kontinuum w przestrzeni C_I.

РЕЗЮМЕ

Доказано, что множество решений включения $x' \in F(t, x)$ удовлетворяющих начальному условию $x(0) = x_0$, где E полунепрерывная сверху многозначная функция с компактными выпуклыми значениями, представляет континуум, в пространстве C_I.

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