ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXVIII, 11 SECTIO A 1984

Instytut Matematyki
Uniwersytet Łódzki

L. MIKOŁAJCZYK, J. NOWAK

Investigation of Selected Extremal Problems in the Space of Univalent Functions in a Half-plane

Badanie pewnych problemów ekstremalnych w klasie funkcji jednolistnych w półpłaszczyźnie

Исследование некоторых экстремальных задач в классе функций однолистных в полуплоскости

1. Introduction. In the paper there has been presented an attempt of applying the methods of complex analysis and optimal control to investigations of extremal problems for holomorphic and univalent functions in a half-plane. The expedience of investigating extremal problems in the space of such functions is justified, among other things, by their importance in questions of the mechanics of continuous media.

2. Preliminary notes. Let H^1 stand for a class of functions f holomorphic and univalent in the upper half-plane $P_z^+ = \{z : \operatorname{Im} z > 0\}$, which transform this half-plane onto domains contained in the half-plane $P_w^+ = \{w : \operatorname{Im} w > 0\}$. Besides, these functions are normalized by the condition

$$(1.1) \qquad \lim_{P_z^+ \ni z \to \infty} [f(z) - z] = 0.$$

The class H^1 is not empty since the identity function belongs to it. In the paper [5] it was proved that H^1 is connected in P_z^+ and is not a compact class.

In the paper [2] a possibility of using the variational-parametric method to the examination of extremal problems in the class H^1 was shown. The idea of the method rests upon a simultaneous examination of two differential equations satisfied by some function $F(w, \tau)$ which is strictly connected with an extremal function f corresponding to the problem considered. In one of these equations there occurs the derivative of the function F with respect to the variable w, while in the other equation – the derivative of F with respect to a real parameter $\tau \in [0, t_0]$ where t_0 is some fixed number. The first equation is obtained on the basis of the variational method, the other – on the basis of Löwner's parametric theory [6] applied to functions of class H^1. Hence we derive the name of the variational-parametric method.

In papers [1], [3] and [5] the variational-parametric method is applied to investigating extremal problems in some subclasses of functions of class H^1.

Let H^1_L denote the set of all functions $f \in H^1$ for which the complement of $f(P_z^+)$ to P_w^+ is bounded. In virtue of the Schwarz symmetry principle each function $f \in H^1$, has an analytic continuation in $P_z^- = \{z : \operatorname{Im} z < 0\}$ according to the formula $f(z) = \overline{f(\overline{z})}$, $z \in P_z^-$. Moreover, by continuity, the function $f \in H^1_L$ is uniquely continuable onto the entire real axis $\partial P_z^+ = \{z : \operatorname{Im} z = 0\}$, with the exception of some bounded part of ∂P_z^+. The function f thus continued is holomorphic in some ring

$$K_R = \{z : R < |z| < \infty, R > 0\}.$$

Expanding f in K_R in a Laurent series, we shall obtain

$$f(z) = z + \sum_{k=1}^{\infty} c_k z^{-k} \tag{1.2}$$

where all the coefficients c_k are real numbers.

Let $\widetilde{H}^1_L$ stand for the set of all functions $f \in H^1_L$, each of which transforms the half-plane P_z^+ onto a domain $G_0 = f(P_z^+)$ of the complex plane C_w, obtained form the half-plane P_w^+ by the removal of a finite number of pairwise disjoint Jordan arcs. One can prove that the class $\widetilde{H}^1_L$ is non-empty and is dense in H^1_L in the sense of the topology of uniform convergence inside P_z^+. One can also prove (cf. [2] or [3]):

Theorem A. *Let f belong to H^1 (or to H^1_L or $\widetilde{H}^1_L$). Then there exist three classess $\{f_k(z, t)\}$ $(k = 1, 2, 3)$ of functions $f_k(z, t)$ of class H^1 (or of H^1_L or $\widetilde{H}^1_L$, respectively) depending on a real parameter t and having, with $z \in P_z^+$ and with small $t > 0$, the representation*

$$f_k(z, t) = f(z) + tQ_k(z, f) + o(z, t), \; k = 1, 2, 3,$$

where

$$Q_1(z, f) = \frac{f'(z)}{\lambda - z},$$

$$Q_2(z, f) = Q_2(z, f; w_0) = \frac{A}{f(z) - w_0} + \frac{\overline{A}}{f(z) - \overline{w}_0},$$

$$Q_3(z, f) = Q_2(z, f; f(\zeta)) + f'(z)\left(\frac{A}{(f'(z))^2(\zeta - z)} + \frac{\overline{A}}{(\overline{f'(z)})^2(\overline{\zeta} - z)}\right);$$

A and ζ are arbitrary complex numbers ($\zeta \in P_z^+$), w_0 is any point from the half-plane P_w^+, external for the domain $G_0 = f(P_z^+)$, λ is an arbitrary real number, $o(z, t)$ is a holomorphic function in P_z^+ such that $t^{-1}\, o(z, t) \to 0$ uniformly inside P_z^+ and in the neighbourhood of infinity as $t \to 0$.

In particular, if f is any function of class $\tilde{H}_L^1$, then, to this class, also belongs a function given by the formula

$$f_*(z) = f(z) + t\left(\frac{A}{f(z) - w_0} + \frac{\bar{A}}{f(z) - \bar{w}_0}\right).$$

This formula is used to some characterization of the set G_0 according to theorem A.

Theorem B. *For any function $f \in \tilde{H}_L^1$, $f(z) \neq z$, $z \in P_z^+$, there exist: a number $t_0 > 0$, real functions $u_k = u_k(t)$ $(k = 1, 2, \ldots, m,\ m \geqslant 1)$ piecewise continuous on the interval $[0, t_0)$, with no points of discontinuity of the second kind, and non-negative functions $\delta_k = \delta_k(t)$ $(k = 1, \ldots, m)$, $0 \leqslant t \leqslant t_0$, $\delta_1 + \ldots + \delta_m = 1$, such that $f(z) = \Phi(z, 0)$ where $\Phi(z, t) \in \tilde{H}_L^1$, $t \in [0, t_0]$, and the function $z = F(w, t)$ inverse to a function $w = \Phi(z, t)$ is a solution of the equation*

$$\frac{\partial F(w, t)}{\partial t} = \sum_{k=1}^{m} \frac{\delta_k(t)}{u_k(t) - F(w, t)},$$

satisfying the condition $F(w, t_0) = w$, $w \in P_w^+$. It turns out that the function $F(w, t)$ satisfies also the condition $F(f(z), 0) = z$.

The function $F(w, t)$ is called a function associated with the function f. The equation occurring in Theorem B bears the name of the Löwner equation for a half-plane:

In particular $(m = 1)$, the function $F(w, t)$ can be obtained as the integral of the equation

$$\frac{\partial F(w, t)}{\partial t} = \frac{1}{u(t) - F(w, t)}, \quad 0 \leqslant t \leqslant t_0,$$

with the condition $F(w, t_0) = w$.

Remark. Normalization condition (1.1) secures the uniqueness of the inverse function (cf. [2], p. 143).

It is to be proved that the function $\Phi(z, t)$ (in the special case under consideration) satisfies the equation

$$\frac{\partial \Phi(z,t)}{\partial t} + \frac{1}{u(t)-z}\,\frac{\partial \Phi(z,t)}{\partial z} = 0$$

and the condition

$$\Phi(z,0) = f(z).$$

3. Of late years, there appeared many papers (comp. e.g. [8], [9], [10]) in which some extremal problems were investigated by means of: variational methods, parametric one and that of optimal control. As has been indicated in the introduction, the present paper constitutes – in the author's opinion – the first tentative of applying the above-mentioned methods to the solving of a concrete extremal problem concerning a selected class of holomorphic and univalent functions in a half-plane.

Theorem 2.1. *Let the differential equation*

(2.1) $$\frac{\partial f(z,t)}{\partial t} + \frac{1}{u(t)-z}\,\frac{\partial f(z,t)}{\partial z} = 0$$

be given, where $(z,t) \in P_z^+ \times [0, t_0]$, while $u(t)$ is a measurable function on the interval $[0, t_0]$. If $\mathcal{C}(z)$ is a holomorphic function in the half-plane P_z^+, and $f(z,t)$ is a solution of equation (2.1) with the initial condition

(2.2) $$f(z,0) = \mathcal{C}(z),$$

then there exists exactly one holomorphic and univalent function $g : P_z^+ \times [0, t_0] \to P_z^+$ such that

(2.3) $$f(z,t) = \mathcal{C}(g(z,t)) \text{ for } t \in [0, t_0].$$

Proof. Let us take into consideration the equation

(2.4) $$\frac{\partial g(z,t)}{\partial t} + \frac{1}{u(t)-z}\,\frac{\partial g(z,t)}{\partial z} = 0$$

with the initial condition $g(z,0) = z$.

In this way we define, for all $t \in [0, t_0]$, some holomorphic function $g(z,t)$ of the variable z. Besides, the equation

$$g(z,t) = a, \ \operatorname{Im} a > 0,$$

defines, for t sufficiently small, a curve $z(t;a)$ such that

(2.5) $$g(z(t;a),t) = a.$$

This means that the curves $z(t;a)$ are the characteristics of equation (2.4) and satisfy the differential equation

$$\frac{dz(t;a)}{dt} + \frac{1}{u(t) - z(t;a)} = 0,\ z(0;a) = a.$$

Since $\operatorname{Im} u(t) = 0$, $\operatorname{Im} z(t;a) > 0$, therefore

(2.6) $$\operatorname{Im} \frac{dz(t;a)}{dt} > 0$$

Hence we infer that

$$\operatorname{Arg} \frac{dz(t;a)}{dt} \in (0, \pi).$$

By taking account of (2.5) and (2.6), it can be verified that the function $g(z,t)$ is, for all $t \in [0, t_0]$. a univalent function in the half-plane P_z^+ and transforms it onto its certain subset.

Since $g(z,t)$ is a solution of equation (2.4), so is also each function

(2.7) $$f(z,t) = \mathcal{C}(g(z,t)).$$

Moreover, since $g(z, 0) = z$, therefore $f(z, 0) = \mathcal{C}(z)$. From this and from the theorem on the uniqueness of solution of a differential equation we deduce that the function $f(z,t)$, appearing in the proposition of our theorem, really has form (2.7), i.e. form (2.3).

On the ground of Löwner's theory, it can be shown that the set of solutions of equation (2.1) is dense in H^1. Whereas on the basis of Theorem 2.1, from the continuity of solution of a differential equation we infer that each function $\mathcal{C} \in H^1$ can be arbitrarily approximated by the solution $f(z,t)$ of equation (2.1) with the initial condition $f(z,0) = \mathcal{C}(z)$. The justification of these facts can be found in monograph [7] where there have been collected, among other things, some basic results concerning Löwner's theory for a half-plane and its applications.

4. We shall now give an example of applying the information gathered in sections 2 and 3 to the solving of a concrete extremal problem.

Let $\mathcal{C} \in \tilde{H}_L^1$, and let it possess an expansion in a Laurent series, given by formula

(1.2). From the considerations contained in sections 2 and 3 it follows that, for this function, there exist: a number $t_0 > 0$ and a real function $u = u(t)$ piecewise continuous on the interval $[0, t_0]$, with no points of discontinuity of the second kind, continuous on the right at the point 0 and on the left at the point t_0, such that $\mathcal{C}(z) = f(z, 0)$ where $f(z, t) \in \tilde{H}^1_L$ and, with any $t \in [0, t_0]$, is the integral of equation (2.1) with condition (2.2). Since, with each $t \in [0, t_0]$, the function $f(z, t) \in \tilde{H}^1_L$, therefore this function has in the neighbourhood of infinity the expansion

$$f(z, t) = z + \sum_{k=1}^{\infty} x_k(t) z^{-k} \tag{3.1}$$

with real coefficients $x_k = x_k(t)$ $(k = 1, 2, \ldots)$.
Substituting (3.1) into equation (2.1), we obtain, for the coefficients $x_k(t)$ $(k = 1, 2, \ldots)$, the following system of equations:

$$\dot{x}_1 = 1$$

$$\dot{x}_2 = u$$

$$\dot{x}_3 = u^2 - x_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\dot{x}_k = u^{k-1} - \sum_{m=1}^{k-2} m x_m u^{k-m-2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

for almost all $t \in [0, t_0]$.

If c_k $(k = 1, 2, \ldots)$ are coefficients in the expansion of the function $\mathcal{C}$ in a Laurent series, then from condition (2.2) we have

$$x_k(0) = c_k, \; k = 1, 2, \ldots . \tag{3.2}$$

It can be shown (cf. [7], p. 243) that $c_1 \leqslant 0$, with that $c_1 = 0$ if and only if $\mathcal{C}(z) = z$. From Theorem B one can deduce that the condition $x_k = x_k(t_0) = 0$, $k = 1, 2, \ldots$, holds. From this and from condition (3.2) it follows that $t_0 = -c_1$. This equality is obtained at once from the conditions: $\dot{x}_1 = 0$, $x_1(0) = c_1$, $x_1(t_0) = 0$.

From among all functions of class H^1_L we choose those for which the first two coefficients c_1 and c_2 of the expansion in a Laurent series are known. Let us consider the following extremal problem.

Problem 1. Find the extremal values of the coefficient c_3 in expansion (1.2), with the first two coefficients c_1 and c_2 fixed.

Let a point move in the space of variables (x_1, x_2, x_3) according to the law

$$\begin{gathered} \dot{x}_1 = 1, \dot{x}_2 = u, \dot{x}_3 = u^2 - x_1, \; t \in [0, t_0] \\ x_i(0) = c_i, \; x_i(t_0) = 0, \; i = 1, 2, 3. \end{gathered} \tag{3.3}$$

We assume that the function $u = u(t)$ is a control function. Note that

$$\int_0^{t_0} (u^2(t) - x_1(t))\, dt = \int_0^{t_0} \dot{x}_3(t)\, dt = x_3(t_0) - x_3(0) = -x_3(0) = -c_3,$$

and thus,

$$x_3(0) = c_3 = -\int_0^{t_0} (u^2(t) - x_1(t))\, dt. \tag{3.4}$$

Let us consider.

Problem 2. Determine a control u that carries a point from a position (c_1, c_2, x_3) at the instant $t = 0$ to the position (0, 0, 0) at the instant t_0, so that the functional $J_1 = x_3(0)$ should take the maximal value or, which is equivalent, that the functional

$$J(u) = \int_0^{t_0} (u^2(t) - x_1(t))\, dt$$

should attain the minimal value.
Applying the transformation

$$\tilde{x}_1 = x_1 - t, \tilde{x}_2 = x_2,$$

from (3.3) we obtain

$$\dot{\tilde{x}}_1 = 0, \dot{\tilde{x}}_2 = u, \tilde{x}_1(t_0) = -t_0, \tilde{x}_2(t_0) = 0.$$

(3.5)

$$\tilde{x}_i(0) = c_i, i = 1, 2;$$

$$J(u) = \int_0^{t_0} (u^2(t) - \tilde{x}_1(t) - t)\, dt. \tag{3.6}$$

Then Problem 2 can be reformulated in the following manner.

Problem 3. Determine a control u that carries a point from the position c_1, c_2 at the instant $t = 0$ to the position $(-t_0, 0)$ at the instant t_0, so that functional (3.6) should attain its minimum. Of course, the optimal control found for Problem 3 is optimal for Problem 2.
Let us put

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then (3.5) can be written in the form

$$\dot{\tilde{x}} = A(t)\tilde{x} + B(t)u$$

where

$$A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$J(u) = \int_0^{t_0} (u^2(t) - \tilde{x}_1(t) - t)\, dt.$$

The matrices A(t) and $B(t)$ as constants are continuous. We introduce the notations

$$f^0(t, \tilde{x}) = -\tilde{x}_1(t) - t,$$

$$h^0(t, u) = u^2(t).$$

Note that the function f^0, with a fixed t, is a convex and bounded function with $t \in [0, t_0]$. In turn, the function h^0 is a sharply convex function, and

$$|h^0(t, u)| \geqslant |u|^2.$$

Making use of the theorems given in monograph [4] on pages 226–234, we find that Problem 3 does possess a solution. Moreover, the control $u^*(t)$ together with the trajectory $x^*(t)$ will be optimal in our problem if and only if there exists a vector

$$\eta(t) = [\eta_0, \eta(t)]$$

such that

$$\dot{\eta}_0 = 0, \eta_0 < 0,$$

$$\dot{\eta} = -\eta_0 \frac{\partial f^0(t, x)}{\partial x} - \eta A(t),$$

and such that almost everywhere on the interval $[0, t_0]$ the relation

$$\eta_0 h^0(t, u^*(t)) + \eta(t) B(t) u^*(t) = \max_u [\eta_0 h^0(t, u) + \eta(t) B(t) u]$$

is satisfied.

We determine successively

$$\eta(t) = [\eta_0 t + d_1, d_2], u^*(t) = -(d_2 / 2\eta_0)$$

and

$$x^*(t) = \begin{bmatrix} d_3 \\ -\dfrac{d_2}{2\eta_0} t + d_4 \end{bmatrix},$$

where d_1, d_2, d_3, d_4 are fixed real numbers. Since

$$\tilde{x}_1^*(0) = c_1^*, \tilde{x}_2^*(0) = c_2^*, \tilde{x}_3^*(0) = c_3^*, \tilde{x}_1^*(t_0) = -t_0, \tilde{x}_2^*(t_0) = \tilde{x}_3^*(t_0) = 0,$$

therefore from (3.6), the above relations and the fact that

$$\tilde{x}_3^*(t) = -(1/2)t^2 + \left(\frac{d_2^2}{4\eta_0^2} - d_3\right)t + d_5, \; d_5 = \text{const},$$

we shall easily find the relation

$$c_3^* = -\frac{c_2^{*2}}{c_1^*} - \frac{1}{2}c_1^{*2}.$$

Hence we infer that if the function $f \in \tilde{H}_L^1$ has expansion (1.2) with c_1 and c_2 fixed, then the sharp estimate

$$c_3 \leqslant \frac{c_2^2}{c_1} - \frac{1}{2}c_1^2 \tag{3.7}$$

takes place. Estimate (3.7) was obtained, in some other way, by V. V. Sobolev and T. N. Sellyakhova in 1974 [11].

The result obtained constitutes a confirmation of the efficacy of simultaneous application of methods of complex analysis and optimal control to investigating extremal problems in the spaces of holomorphic and univalent functions in a half-plane.

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STRESZCZENIE

W pracy tej przedstawiono próbę stosowania metod analizy zespolonej i sterowania optymalnego do badania problemów ekstremalnych dla funkcji holomorficznych i jednolistnych w półpłaszczyźnie. Celowość badania problemów ekstremalnych dla takich funkcji jest uzasadniona ich zastosowaniami w mechanice ośrodków ciągłych.

РЕЗЮМЕ

В работе представлена попытка использования метода комплексного анализа и теории оптимального управлуния в изучении экстремальных задач для отображений голоморфных и однолистных в полуплоскости. Целесообразность изучения экстремальных задач для таких отображений обоснованна возможностью их применения в механике непрерывных сред.