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The Topological Degree and Fixed Point Theorem for 1-Set Contractions

Stopień topologiczny i twierdzenie o punkcie stałym dla wielowartościowych odwzorowań nieoddalających

Топологический индекс и теоремы о неподвижной точке для многозначных отображений удовлетворяющих условию Липшица с константой 1

In the papers [5] and [7] the topological degree for maps of the form $I - T$, where T is set-valued analogue of the „limit compact" mappings of Sadowski [6], is mentioned. We extend this notion to the multivalued 1-set contractions.

1. Condensing maps and 1-set contractions. Let G be an open bounded subset of a Banach space X.

Definition 1. A mapping $T: G \to 2^X$ is said to be upper semicontinuous (USC) at $x_0 \in G$ if the set $T(x_0)$ is closed and convex and for any $\epsilon > 0$ there exists $\delta > 0$ such that $T(x) \in B(T(x_0), \epsilon)$ for $x \in B(x_0, \delta)$.

By $B(T(x_0), \epsilon)$ we mean ϵ-neighbourhood of the set $T(x_0)$, i.e. the sum of the balls $B(q, \epsilon)$, $q \in T(x_0)$.

T is called USC on G if it is USC at each point of G.

Definition 2. The measure of noncompactness of a bounded set $D \subset X$ is defined as follows:

$$\alpha(D) = \inf \; \gamma > 0: \text{there exist sets } B_1, \ldots, B_n \subset X \text{ such that } \cup B_i \supset D, \; \delta(B_i) < \gamma$$
$$\text{for } i = 1, \ldots, n \; .$$

where

$$\delta(B_i) = \sup_{x,y \in B_i} \| x - y \|$$

(comp. Kuratowski [3]).

Definition 3. The USC mapping $T: G \to 2^X$ such that $T(G)$ is bounded is called:

1) k-set contraction ($k \geq 0$) if for every $D \subset G$ satisfies the condition $\alpha(T(D)) \leq \leq k\alpha(D)$,

2) condensing mapping if for each subset D of G with $\alpha(D) > 0$ we have $\alpha(T(D) < < \alpha(D)$.

For multivalued mappings $S, T: G \to 2^{\bar{X}}$ and a scalar a we introduce the operations:

$$(S + T)(x) = \{y + z: y \in S(x), z \in T(x)\}, \ (aS)(x) = \{ay: y \in S(x)\}.$$

Using the degree theory for k-set contractions ($k < 1$) we will show important fact, very usefull in further considerations.

Theorem 1. *Let T_i be a multivalued k_i-set contraction for $k_i < 1$, $T_i: G \to 2^X$, $i = 0, 1$, which satisfies the assumptions:*

$$T_i(G) \text{ is bounded and } x \bar{\in} T_i(x) \text{ for any } x \in \partial G.$$

Let $H: \bar{G} x\ [0, 1] \to 2^X$ be a segment homotopy between T_0 and T_i, i.e.

$$H(x, t) = tT_1(x) + (1 - t)T_0(x), x \in \bar{G}, t \in [0, 1].$$

Assume that

$$x \bar{\in} H(x, t) \text{ for } (x, t) \in \partial G \times [0, 1].$$

Then deg $(I - T_0, G, 0) =$ deg $(I - T_1, G, 0)$, *where* deg *denotes degree in the sense of* [7], *I is the identity mapping.*

Proof. According to Theorem 3 Webb [7] it is sufficiently to show that H is USC mapping (it is easy to see) and that $G_\infty = G_\infty(H)$ is compact (possibly empty), where

$$G_1 = \overline{\text{co}}\,(H(\bar{G} \times [0, 1])), \quad G_n = \overline{\text{co}}\,(H(\bar{G} \cap G_{n-1}) \times [0, 1]))$$

and

$$G_\infty = \bigcap_{n=1}^{\infty} G_n .$$

Let $k = \max(k_0, k_1)$. We can show

$$\alpha(H(\bar{G} \times [0, 1])) \leq k\alpha(G)$$

and using the mathematical induction

$$\alpha(G_{n+1}) = \alpha(H((\bar{G} \cap G_n) \times [0, 1])) \leq k^{n+1}\, \alpha(G) \to 0 \text{ as } n \to \infty.$$

Hence G_∞ is compact since it is closed.

2. Topological degree for 1-set contractions. For $A, B \subset X$ we define

$$d^*(A, B) = \sup_{a \in A}\ d(a, B) = \sup_{a \in A}\ \inf_{b \in B}\ \|a - b\|.$$

Let T be a 1-set contraction, $T: \bar{G} \to 2^X$, and let

$$\eta = d(0, (I - T)(\partial G)) > 0.$$

Choose strict set contraction $\bar{T}$ such that

(1) $$d^*(\bar{T}(x), T(x)) \leq (1/3)\,\eta \text{ for } x \in \bar{G}.$$

For example condition (1) is satisfying if $\bar{T} = tT$, where $1 - t > 0$ is sufficiently small (because $T(\bar{G})$ is bounded).

It is easy to show that

(2) $$d(0, \overline{(I - \bar{T})(\partial G)}) \geq (2/3)\eta.$$

Moreover we obtain

(3) $$d(0, (I - \bar{T})(\partial G)) \geq (2/3)\eta$$

since $I - \bar{T}$ is closed.

Definition 4. Let T be a 1-set contraction and

$$d(0, \overline{(I - T)(\partial G)}) > 0.$$

We define the topological degree of T as follows:

$$\deg(I - T, G, 0) = \deg(I - \bar{T}, G, 0),$$

where $\bar{T}$ is a strict set contraction satysfying (1) and the right hand side denotes degree in the sense of [7].

Lemma 1. *Definition 4 is independent of the choise of* $\bar{T}$.

It follows from theorem 1.

Theorem 2. *Let* $T: \bar{G} \to 2^X$ *be a 1-set contraction and* $0 \notin \overline{(I - T)(\partial G)}$.
Then the above defined degree has the following properties:

a) *If T is a strict set contraction then* deg $(I - T, G, 0)$ *from definition 4 is the same as for strict set contractions.*

b) *If* $(I - T)(\bar{G})$ *is closed and* deg $(I - T, G, 0) \neq 0$, *then there exists* $x \in G$ *such that* $x \in T(x)$.

c) *If* G_1, G_2 *are open sets,* $\bar{G}_1 \cup \bar{G}_2 = \bar{G}$, $G_1 \cap G_2 = \emptyset$ *and* $0 \notin \overline{(I - T)(\partial G_i)}$ *for* $i = 1, 2$, *then*

$$\deg (I - T, G, 0) = \deg (I - T, G_1, 0) + \deg (I - T, G_2, 0).$$

d) *Let* $h: [0, 1] \to \{I - T: T \text{ is a 1-set contraction}\}$ *be a continuous mapping in the following sense: for all* $t \in [0, 1]$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_{x \in \bar{G}} d^*(h(t'), h(t)(x)) < \epsilon \text{ for } t' \in [0, 1], \ |t' - t| < \delta .$$

Suppose that $0 \notin \overline{h(t)(\partial G)}$ *for all* $t \in [0, 1]$. *Then* deg $(h(t), G, 0) = \text{const}\,(t)$.

Proof. We will show only d). For this proof it is sufficient to verify, that if $T: G \to 2^X$ is a 1-set contraction, $\eta = d(0, \overline{(I - T)(\partial G)}) > 0$, $S: G \to 2^X$ is a 1-set contraction such that $I - S \in U(T, (\eta/4))$, where

$$U(T, r) = \{I - S : S \text{ is 1-set contraction and } \sup_{x \in \bar{G}} d^*((I - S)(x), (I - T)(x)) < r\},$$

then

$$\deg (I - T, G, 0) = \deg (I - S, G, 0). \tag{4}$$

In fact, it is easy to check, that for $I - S \in U(T, (\eta/4))$ and $\lambda \in (0, 1)$ with $1-\lambda$ sufficiently small, we have $I - \lambda T, I - \lambda S \in U(T, (\eta/2))$. By definition 4 it is

$$\deg (I - T, G, 0) = \deg (I - \lambda T, G, 0) \text{ and } \deg (I - S, G, 0) = \deg (I - \lambda S, G, 0).$$

We will prove the equality of degrees of $I - \lambda T$ and $I - \lambda S$. Consider the mapping

$$H(t)(x) = t(I - \lambda T)(x) + (1 - t)(I - \lambda S)(x).$$

Using the property

$$d^*(aA + bB, a\tilde{A} + b\tilde{B}) \leqslant |a|\, d^*(A, \tilde{A}) + |b|\, d^*(B, \tilde{B})$$

we can show that $H(t) \in U(T, (\eta/2))$ so that $0 \notin \overline{H(t)(\partial G)}$. Hence, in view of theorem 1, we obtain

$$\deg(I - \lambda T, G, 0) = \deg(I - \lambda S, G, 0)$$

and so (4) is true.

3. The fixed point theorem. Theorem 3. *Let G be an open subset of a Banach space X and $T: \bar{G} \to 2^X$ be* USC *1-set contraction such that $(I - T)(\bar{G})$ and $(I - T)(\partial G)$ are closed. Suppose that there exists $w \in G$ with*

(5) $$T(x) - w \not\ni m(x - w) \text{ for } x \in \partial G,\ m > 1.$$

Then there exists $x \in \bar{G}$ such that $x \in T(x)$.

Proof. According to theorem 2 we have to show that the homotopy

$$H(t)(x) = (I - tT)(x) + (1 - t)w,\ t \in [0, 1],\ x \in \bar{G},$$

satisfies the condition

$$0 \notin \overline{H(t)(\partial G)} \text{ for all } t \in [0, 1].$$

But the sets $H(t)(\partial G)$, $t \in [0, 1]$, are closed: $H(1)(\partial G) = (I - T)(\partial G)$ by assumption, and $H(t)(\partial G)$, $t \in [0, 1)$ because tT is strict set contraction.

Hence it is sufficient to check that $0 \notin H(t)(\partial G)$.

1) If $0 \in H(0)(x)$ for $x \in \partial G$ then $0 = w - x$. It is impossible by $w \notin \partial G$.

2) If $0 \in H(t)(x)$, $t \in [0, 1)$, $x \in \partial G$, then $0 \in H(t)(x) = x - tT(x) - (1 - t)w$. Hence $1/t(x - w) \in T(x) - w$. Contradiction with (5).

3) The case $0 \in H(1)(x)$ for $x \in \partial G$ may be omitted since it implies that T has fixed point on ∂G.

Finally, from theorem 2 we obtain

$$\deg(I - T, G, 0) = \deg(I - w, G, 0) = 1$$

and there is $x \in G$ such that $x \in T(x)$.

REFERENCES

[1] Cellina, A., Lasota, A., *A new approach to the definition of topological degree for multivalued mappings*, Atti Accad. Rend. 47 (1969), 434–440.

[2] Lloyd, N. G., *Degree Theory*, Cambridge University Press, Cambridge 1978.

[3] Kuratowski, K., *Sur les espaces complete*, Fund. Math. 15 (1930), 301–309.

[4] Nussbaum, R. D., *The fixed point index and asymptotic fixed point theorem for k-set contractions*, Bull. Amer. Math. Soc. 75 (1969), 490–495.

[5] Petryshyn, W. V., Fitzpatrick, P. M., *A degree theory, fixed point theorem and mapping theorems for multivalued noncompact mappings*, Trans. Amer. Math. Soc. 194 (1974), 1–25.

[6] Sadowski, B. N., *On a fixed point principle*, Funct. Anal. Appl. 1 (1967), 74–76.

[7] Webb, J. R. L., *On degree theory for multivalued mappings and applications*, Boll. Un. Mat. Ital. (5) 9 (1974), 137–158.

STRESZCZENIE

W pracy tej zdefiniowano topologiczny stopień odwzorowania dla nieoddalających odwzorowań wielowartościowych.

РЕЗЮМЕ

В данной работе конструируется типологический индекс для многозначных отображений удовлетворяющих условию Липшица с константой 1.