ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXVIII, 7 SECTIO A 1984

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Fixed Points via Proximity Maps

Punkty stałe a projekcje metryczne

Неподвижные точки и метрические проекции

1. Introduction. Throughout this paper X will denote a (real) Hilbert space, $\mathcal{A}(X)$, $\mathcal{B}(X)$ and $\mathcal{C}(X)$ will denote respectively the family of all (nonempty) closed bounded, convex closed bounded and compact subsets of X.

In the following we denote by $\overline{\text{co}}\, A$ the convex closure of a set A, by dist (x, A) the distance between a point x and a set A, by $H(A_1, A_2)$ the Hausdorff distance between two sets A_1 and A_2:

$$H(A_1, A_2) = \text{Max} \{\text{Sup}\,[\text{dist}\,(a_1, A_2) : a_1 \in A_1]\,;\, \text{Sup}\,[\text{dist}\,(a_2, A_1) : a_2 \in A_2]\} .$$

Finally we denote by P_A the metric projection into A, i.e. for x in X, $P_A(x) = \{y \in A : \| x - y \| = \text{dist}\,(x, A)\}$ and for A_1 in $\mathcal{A}(X)$ $P_A(A_1) = \cup \{P_A(x) : x \in A_1\}$. We recall that a set C is said Chebyshev with respect to $\overline{\text{co}}\, C$ when $P_C(x)$ is a singleton for each x in $\overline{\text{co}}\, C$.

The aim of the present paper is to prove that any (nonempty) bounded subset C of X which is Chebyshev w.r.t. $\overline{\text{co}}\, C$ has the fixed point property for nonexpansive multi-applications with values in $\mathcal{B}(X)$ or in $\mathcal{C}(X)$, under the assumption that the boundary ∂C of C is mapped into C.

2. Preliminary results. In this section we list some well known results and their consequences. Proofs may be found in [2] or [15].

Let $K \in \mathcal{B}(X)$ and let us denote P_K by P. K is Chebyshev w.r.t. X and

(2.1) $$Px = x \iff x \in K.$$

(2.2) $$x \notin K \Rightarrow Px \in \partial K.$$

(2.3) $$y \in K \Rightarrow (x - Px, y - Px) \leq 0$$

for each $x \in X$, hence

(2.4) $$y \in K \Rightarrow \| x - y \|^2 \geq \| x - Px \|^2 + \| Px - y \|^2 .$$

(2.5) $$\| Px - Py \| \leq \| x - y \|$$

for any x, y in X, i.e. P is nonexpansive and , of consequence

(2.6) $$H(P(A), P(B)) \leq H(A, B)$$

for any $A, B \in \mathcal{A}(X)$.

(2.7) Let $T : X \to \mathcal{A}(X)$ be nonexpansive and consider $F = P \cdot T$: F is nonexpansive too, but usually it does not assume closed values. Nevertheless

$$\operatorname{Inf} \left\{ \operatorname{dist}(x, Fx) : x \in K \right\} = 0.$$

In fact the map $G_\alpha : K \to \mathcal{A}(K)$ defined by $G_\alpha(x) = (1 - \alpha)x_0 + \alpha\overline{F(x)}$ is a contraction, for x_0 fixed in K and for any α in $(0, 1)$. G_α has a fixed point ([14] theorem 5), say x_α. We have $x_\alpha = (1 - \alpha)x_0 + \alpha y_\alpha$ with $y_\alpha \in \overline{F(x_\alpha)}$ and then $\operatorname{dist}(x_\alpha, Fx_\alpha) \leq \| x_\alpha - y_\alpha \| = (1 - \alpha) \| x_0 - y_\alpha \| \leq (1 - \alpha) \operatorname{diam} K$.

(2.8) Suppose now that the above nonexpansive T has values in $\mathcal{C}(X)$: in this case $F = P \cdot T$ takes values in $\mathcal{C}(K)$ ([13], 9.6) so F has a fixed point in K ([8] theorem 3.2).

(2.9) We conclude our preparation recalling the following result, which holds in more general spaces than Hilbert ones ([12] theorems 2 and 4, see also [7], [9] and [10]):

Let K be a weakly compact subset of X and T be a nonexpansive mapping with values in $\mathcal{C}(X)$ [or in $\mathcal{B}(X)$, and let X be separable]. T has a fixed point in K if and only if

$$\operatorname{Inf} \quad \operatorname{dist}(x, Tx) : x \in K \quad = 0.$$

3. Main results. Theorem 1. *Let C be a bounded Chebyshev set w.r.t. $\overline{\text{co}}\, C$ and let $T : X \to \mathcal{C}(X)$ be nonexpansive. If $T(\partial C) \subset C$, then T has a fixed point in C.*

Theorem 2. *Let X be separable, C be a bounded Chebyshev set w.r.t. $\overline{\text{co}}\, C$ and $T : X \to \mathcal{B}(X)$ be nonexpansive. If $T(\partial C) \subset C$, then T has a fixed point in C.*

In the following proofs we put $K = \overline{\text{co}}\, C$ and $F = P_K \cdot T$.

Proof of Theorem 1. In view of (2.8) F has a fixed point in K, say x. We claim that $P_C(x)$ is a fixed point of T. This is trivial if $x \in \partial C$ (indeed $Fx = Tx$). If x is an inner point of C, Fx is the union of $Tx \cap K$ with a subset of ∂K, so $x \in Fx$ and $x \in \text{int}\, C \Rightarrow$ $\Rightarrow x \in Tx$. Suppose now $x \in K \setminus C$: let $y = P_C(x)$ and choose a z in Ty such that $\|x - z\| =$ $= \text{dist}\,(x, Ty)$. As $y \in \partial C$, $Ty = Fy \subset C$ and

$$\| x - z \| = \text{dist}\,(x, Fy) \leqslant H(Fx, Fy) \leqslant \| x - y \| = \text{dist}\,(x, C).$$

Unicity of y implies $y = z$.

Proof of Theorem 2. It is sufficient to prove that

$$\text{Inf}\,\left\{\text{dist}\,(x, Tx) : x \in K\right\} = 0.$$

If so indeed, T has a fixed point in K, whose projection into C is still a fixed point of T, by the same argument used in theorem 1.

In view of (2.7) it is possible to construct two sequences $\{x_n\}$ and $\{y_n\}$, $x_n \in K$ and $y_n \in Tx_n$ such that

(3.1) $$\| x_n - P_K\, y_n \| \to 0.$$

Suppose, by contradiction, that

(3.2) $$\text{Inf}\,\{\text{dist}\,(x, Tx) : x \in K\} = \delta > 0.$$

We have

$$\| x_n - y_n \| \geqslant \text{dist}\,(x_n, Tx_n) \geqslant \delta$$

so

$$\text{dist}\,(y_n, K) = \| y_n - P_K\, y_n \| = \| y_n - x_n \| + o(1)$$

hence lim inf dist $(y_n, K) \geqslant \delta$ which means that y_n has to be (for large n) out of K, and (by (2.2) and (3.1)) that x_n has to approach the boundary of K: we may suppose, by the continuity of F, that $x_n \in \partial K$.

For each n let z_n and u_n be the points which are uniquely determined by

$$z_n = P_C\, x_n ; u_n \in Tz_n \text{ such that } \| y_n - u_n \| = \text{dist}\,(y_n, Tz_n).$$

As $u_n \in C$ we have

(3.3) $$\| u_n - x_n \| \geqslant \| z_n - x_n \|$$

moreover

$$(3.4) \qquad \|y_n - u_n\| = \operatorname{dist}(y_n, Tz_n) \leqslant H(Tx_n, Tz_n) \leqslant \|x_n - z_n\|.$$

Finally, from (2.4)

$$(3.5) \qquad \|y_n - u_n\|^2 \geqslant \|y_n - P_K y_n\|^2 + \|P_K y_n - u_n\|^2 =$$

$$= \|y_n - x_n\|^2 + \|x_n - u_n\|^2 + o(1).$$

The combination of (3.4), (3.5), (3.3) and (3.2) leads us to

$$\|x_n - z_n\|^2 \geqslant \|y_n - u_n\|^2 \geqslant \|y_n - x_n\|^2 + \|x_n - u_n\|^2 + o(1) \geqslant$$

$$\geqslant \delta^2 + \|z_n - x_n\|^2 + o(1)$$

a contradiction if $\delta > 0$.

4. Remarks.

1. Suppose T is a single-valued map: we obtain a result of K. Goebel and R. Schöneberg [6]; other results under weaker assumptions on C may be found in [4] and [5].

2. If C is a convex set, theorem 1 is known true even if X is an Opial space ([1], theorem 2), moreover the condition $T(\partial C) \subset C$ may be weakened in the inwardness condition. In this framework the above result holds when C is star-shaped too, but C must be at least weakly compact, while theorem 1 does not require this property.

3. Few results are known for nonexpansive mappings with, as in theorem 2, convex closed bounded values. Usually the set C in which fixed points are searched is assumed to be convex. We only know a result ([12], theorem 3) which holds for nonconvex C, but it requires its weak compactness. Observe that in theorem 2 no assumption of weak compactness is done.

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STRESZCZENIE

W pracy tej wykazano, że dowolny ograniczony podzbiór *C* przestrzeni Hilberta będący zbiorem Czebyszewa względem swej otoczki wypukłej ma własność punktu stałego dla wieloznacznych odwzorowań nieoddalających o wartościach ze zbioru zwartego lub ze zbioru domkniętego, ograniczonego i wypukłego, takich, że brzeg *C* przechodzi w *C*.

РЕЗЮМЕ

В работе доказывается, что произвольное, ограниченное подмножество *C* гильбертого пространства, являющиеся множеством Чебышева относительно своей выпуклой оболочки, обладает свойством неподвижной точки для многозначных отображений (удовлетворяющих условию Липшина с константой равной единице) примающих значения из некоторого компакта или из некоторого, ограниченного и замкнутого множества, при условии, что граница *C* отображается в *C*.