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Pointwise Bounded Families of Holomorphic Functions

Punktowo ograniczone rodziny funkcji holomorficznych

Точечно ограниченные семейства голоморфных функций

Let $\mathfrak{F}$ be a family of functions f holomorphic in a given region (nonempty connected open set [1; p. 57] G of the complex plane The following definitions are quite familiar).

Definition 1. The family $\mathfrak{F}$ is said to be pointwise bounded in G if for each $z \in G$ there exists a finite number $M(z)$ such that $|f(z)| \leq M(z)$ for each $f \in \mathfrak{F}$.

Definition 2. The family $\mathfrak{F}$ is said to be locally bounded in G if for each $\zeta \in G$ there exists a neighbourhood $N_\zeta \subset G$ of ζ and a positive number $K = K(N_\zeta)$ such that $|f(z)| \leq K$ for all $z \in N_\zeta$ and every $f \in \mathfrak{F}$.

It is easily verified that the local boundedness of the family $\mathfrak{F}$ in the region G is equivalent to the uniform boundedness of $\mathfrak{F}$ on compact subsets of G which is the basic notion in the theory of normal families of holomorphic functions.

On page 97 of [2] it is claimed, and we quote: *Ist $\mathfrak{F}$ punktweise beschränkt so gibt es zu jeder kompakten Teilmenge K von G eine endliche Konstante M = M(K) derart, dass M(z) ≤ M gilt.* This means that if $\mathfrak{F}$ is pointwise bounded in G then the functions in $\mathfrak{F}$ are uniformly bounded on every compact subset of G. For specialists of the subject it is just a lapse on the part of the author but it may be misleading to others.

Although the notion of locally bounded families of holomorphic functions has been discussed in various books (see for example [1], [3]), the fact that a pointwise bounded family $\mathfrak{F}$ of holomorphic functions may not be locally bounded does not appear to have been formally illustrated. It seems desirable to fill this „gap" by giving an explicit example. A non-constant entire function bounded on all rays passing through the origin, which was constructed by Newman [4] may be used for the purpose.

Let f be the entire function of Newman and for $n = 1, 2, \ldots$ define

$$f_n : z \to f(nz), \qquad z \in \mathbb{C}.$$

It is clear that the collection of these functions forms a family $\mathcal{F}$ of entire functions which is pointwise bounded in the complex plane. On the other hand, the family $\mathcal{F}$ is unbounded in each neighbourhood of the origin. Indeed, howsoever small the positive number ϵ may be sequence of disks

$$\{n U_\epsilon\}_{n=1}^{\infty} \quad \text{where} \quad U_\epsilon := \{z \in \mathbb{C} : |z| < \epsilon\}$$

constitutes an exhaustion [1, p. 212] of the complex plane. Since f is a non-constant entire function,

$$\sup_{z \in n U_\epsilon} |f(z)|$$

must tend to $+\infty$ with n.

Remark. We have just shown that for every $\epsilon > 0$,

$$\sup_n (\max_{|z| = \epsilon} |f_n(z)|) = \infty .$$

In fact, we may as well replace $\max\limits_{|z| = \epsilon} |f_n(z)|$ by

$$((1/2\pi) \int_0^{2\pi} |f(\epsilon e^{i\theta})|^p \, d\theta)^{1/p}, \; p > 0$$

which is generally much smaller. For this we simply need to recall that if f is holomorphic in $|z| < R$ then for every $p > 0$ and $0 < r < R$ we have (see for example [5, p. 103])

$$\max_{|z| = r/2} |f(z)| \leqslant 3^{1/p} ((1/2\pi) \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta)^{1/p} .$$

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STRESZCZENIE

W pracy podano prosty przykład rodziny funkcji holomorficznych w obszarze G, która jest wspólnie ograniczona w każdym punkcie tego obszaru, lecz nie jest niemal ograniczona (w sensie Saksa-Zygmunda) w tym obszarze.

РЕЗЮМЕ

В работе приведен простой пример семейства функций которые голоморфны и ограничены в каждой точке области G но не являются почти ограниченными в этой области в смысле Сакса-Зигмунда.