ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXVIII, 4 SECTIO A 1984

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Predictability of Cadlag Processes without Probability

Prognozowalność regularnych procesów bez prawdopodobieństwa

Предсказываемость регулярных процессов без вероятностей

C. Dellacherie and P.-A. Meyer [2; Chap. IV, 69, p. 127] defined the predictable stopping times in a way independent of probability. Using this definition and proceeding along the same main lines as in [2; Chap. IV, 88C], [3; p. XIII–XV] or in [4; Chap. II], it is possible to eliminate all arguments based on probability from the proof of the standard criterion for predictability of cadlag processes. The purpose of the present paper is to explain this possibility.

1. Background.

1.1 Predictable sets and processes. Let $(\Omega, \mathcal{F})$ be a measurable space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $\mathcal{F}_{0-}$ be a distinguished sub-σ-field of the σ-field $\mathcal{F}_0$. According to [2; IV, 67], the corresponding σ-field $\mathcal{P}$ of predictable subsets of $[0,\infty)\times\Omega$ is generated by the family of sets

$$\mathcal{P}_0 = \left\{[t,\infty)\times B : t \geq 0, B \in \mathcal{F}_{t-}\right\},$$

where

$$\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s \quad \text{for } t > 0.$$

Let E be a separable metric space and $\mathcal{B}(E)$ the σ-field of all its Borel subsets. The separability implies that $\mathcal{B}(E^2)$ is equal to the product σ-field $\mathcal{B}(E)\times\mathcal{B}(E)$. An E-valued process $X = (X_t)_{t\geq 0}$ on Ω is simply a mapping

$$X : [0, \infty] \times \Omega \ni (t, \omega) \longrightarrow X_t(\omega) \in E.$$

Such a mapping is called:

1) a predictable process iff it is measurable from $([0, \infty) \times \Omega, \mathcal{P})$ to $(E, \mathcal{B}(E))$,

2) an $(\mathcal{F}_t)$-adapted process iff for each $t \geqslant 0$ the mapping $X_t : \Omega \ni \omega \to X_t(\omega) \in E$ is measurable from $(\Omega, \mathcal{F}_t)$ to $(E, \mathcal{B}(E))$, and

3) a cadlag process iff for each $\omega \in \Omega$ the trajectory $[0, \infty) \ni t \to X_t(\omega) \in E$ is right continuous on $[0, \infty)$ and has left-side limits everywhere on $(0, \infty)$.

1.2. Stopping times and the σ-fields $\mathcal{F}_{\tau-}$. A $[0, \infty]$-valued function τ on Ω is called a stopping time of the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ iff $\tau \leqslant t \quad \in \mathcal{F}_t$ for each $t \geqslant 0$ or, which is the same, iff the stochastic interval

$$[\![\tau, \infty[\![= \{(t, \omega) : \omega \in \Omega, \tau(\omega) \leqslant t < \infty\}$$

is $(\mathcal{F}_t)$-adapted. According to [2; IV, 54.2], for any stopping time τ, the σ-field $\mathcal{F}_{\tau-}$ of subsets of Ω is generated by the family of all sets of the form

$$t \leqslant \tau \quad \cap B, \text{ where } t \geqslant 0 \text{ and } B \in \mathcal{F}_{t-}.$$

It is evident that if τ is identically equal to a finite constant t, then $\mathcal{F}_{\tau-}$ coincides with the $\mathcal{F}_{t-}$ defined in 1.1.

1.2.1. If τ is a stopping time then $\{\tau < \infty\} = \Omega \setminus \bigcap_{n \in \mathbb{N}} \{n \leqslant \tau\} \in \mathcal{F}_{\tau-}$ and the mapping

$$G_\tau : \{\tau < \infty\} \ni \omega \to (\tau(\omega), \omega) \in [0, \infty) \times \Omega$$

is measurable from $(\{\tau < \infty\}, \mathcal{F}_{\tau-})$ to $([0, \infty) \times \Omega, \mathcal{P})$. Indeed, if $B \in \mathcal{F}_{t-}$ then $G^{-1}([t, \infty) \times B) = \{\tau < \infty\} \cap \{t \leqslant \tau\} \cap B$, so that $G^{-1}(P) \in \mathcal{F}_{\tau-}$ for each $P \in \mathcal{P}_0$.

1.2.2. As a consequence, if X is a predictable E-valued process and τ a stopping time, then the mapping

$$X_\tau : \{\tau < \infty\} \ni \omega \to X_{\tau(\omega)}(\omega) \in E$$

is measutable from $(\{\tau < \infty\}, \mathcal{F}_{\tau-})$ to $(E, \mathcal{B}(E))$. In particular, each predictable process is $(\mathcal{F}_t)$-adapted.

1.3. Predictable times and their restrictions. According to the definition introduced in [2; IV, 69], a $[0, \infty]$-valued function τ on Ω is called a predictable time iff $[\![\tau, \infty[\![\in \mathcal{P}$. It follows from 1.2.2 that each predictable time is a stopping time.

1.3.1. If $\mathcal{F}_{0-} = \mathcal{F}_0$ and the measurable space $(\Omega, \mathcal{F})$ carries a probability measure P, such that all P-negligible subsets of Ω belong to $\mathcal{F}_0$, then a stopping time is predictable if and only if it is foretellable, see [2; IV, 71 and 77] or [4; II, T 13]. This equivalence makes predictable times important for theory of stochastic processes.

1.3.2. If τ is a predictable time, then $[\![\tau]\!] \in \mathcal{P}$. Indeed, $[\![\tau]\!] = [\![\tau, \infty[\![\setminus]\!]\tau, \infty[\![$, where $[\![\tau, \infty[\![\in \mathcal{P}$ by definition of the predictable time, and $]\!]\tau, \infty[\![= \bigcup_n [\![\tau + 1/n, \infty[\![\in \mathcal{P}$ since the right shifts of predictable sets are predictable.

The restriction τ_A of a stopping time τ to a set $A \subset \Omega$ is defined by

$$\tau_A(\omega) = \begin{cases} \tau(\omega), \text{ if } \omega \in A, \\ \infty, \text{ if } \omega \in \Omega \setminus A. \end{cases}$$

1.3.3. Lemma. *Let τ be a predictable time and A a subset of Ω. Then τ_A is a predictable time if any only if $\{\tau < \infty\} \cap A \in \mathcal{F}_{\tau-}$.*

Proof. The Lemma is equivalent to [2; IV, 73(c)], the proof of which is based on [2; IV, 67(b)]. Arguing as in the latter,

$$\{\tau < \infty\} \cap A = G^{-1}([\![\tau_A, \infty[\![),$$

so that, by 1.2.1, if τ_A is a predictable time, then $\{\tau < \infty\} \cap A \in \mathcal{F}_{\tau-}$. The proof of the opposite implication, given below, is somewhat more direct then that in [2; IV, 73(c)]. The family Φ_τ of all the sets of the form $[\![\tau_A, \infty[\![$, where $A \subset \Omega$, is a σ-field with the unity $[\![\tau, \infty[\![$, and

$$R_\tau : 2^\Omega \ni A \to [\![\tau_A, \infty[\![\in \Phi_\tau$$

is an epimorphism of the σ-field 2^Ω onto the σ-field Φ_τ. We have to prove that $R_\tau(A) = R_\tau(\{\tau < \infty\} \cap A) \in \mathcal{P}$ whenever $\{\tau < \infty\} \cap A \in \mathcal{F}_{\tau-}$. This will follow, when we show that $R_\tau(A) \in \mathcal{P}$ whenever $A \in \mathcal{F}_{\tau-}$. Since R_τ is a morphism, it is sufficient to prove that $R_\tau(A) \in \mathcal{P}$ for each member A of a family generating the σ-field $\mathcal{F}_{\tau-}$. So, according to 1.1 and 1.2 it remains to verify that $R_\tau(\{t \leqslant \tau\} \cap B) = ([t, \infty) \times B) \cap [\![\tau, \infty[\![\in \mathcal{P}$ whenever $B \in \mathcal{F}_{t-}$.

2. Criterion for predictability. Theorem of P. -A. Meyer [6; VII, T49] is a prototype of the criterion for predictability of cadlag processes which may be found in [1; IV, T31], [2; IV, 88C], [3; p. XIV] and [4; II, 20]. A probability free formulation of this criterion reads as follows.

2.1. Assumption. Let $(\Omega, \mathcal{F})$ be a measurable space with a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ and with distinguished sub-σ-field $\mathcal{F}_{0-}$ of the σ-field $\mathcal{F}_0$. Let E be a separable metric space and let X be an $(\mathcal{F}_t)$-adapted cadlag process on Ω with values in E.

2.2. Theorem. *Under Assumptions 2.1, the process X is predictable if and only if the two conditions are satisfied simultaneously:*

(a) *the set $\{(t, \omega) \in (0, \infty) \times \Omega : X_{t-}(\omega) \neq X_t(\omega)\}$ is contained in a countable sum of graphs of predictable times and*

(b) *for each predictable time τ, the mapping X_τ is measurable from $(\{\tau < \infty\}, \mathcal{F}_{\tau-})$ to $(E, \mathcal{B}(E))$.*

2.3. Formulation involving probability. In addition to Assumptions 2.1 suppose that the measurable space $(\Omega, \mathcal{F})$ carries a probability measure P, such that $\mathcal{F}_{0-}$ contains all

P-negligible subsets of Ω. Then, as it follows at once from [2; IV, 88B] or from the Corollary in our Section 2.4, condition (a) is equivalent to the following condition:

(a′) $P\left\{\tau < \infty \text{ and } X_{\tau-} \neq X_\tau\right\} = 0$ for each totally inaccessible stopping time τ. Replacing (a) with (a′) in Theorem 2.2, we obtain the criterion for predictability in its „classical" version.

2.4. Necessity of condition (a). We shall sketch two proofs. The first proof starts with remark that, by an argument as in [6; IV, 14(b)], for each $\epsilon > 0$ and each n,

$$\tau_n^\epsilon = n\text{-th debut of } \left\{(t, \omega) : \operatorname{dist}(X_{t-}(\omega), X_t(\omega)) > \epsilon\right\}$$

is a wide sense stopping time, so that

$$]\!]\tau_n^\epsilon, \infty[\![\in \mathcal{P}$$

(because the process $1\!\!1_{]\!]\tau_n^\epsilon, \infty[\![}$ is left-continuous and $(\mathcal{F}_{t+}^{\circ})$-adapted). From the equality

$$[\![\tau_n^\epsilon]\!] = \left\{(t, \omega) : \operatorname{dist}(X_{t-}(\omega), X_t(\omega)) > \epsilon\right\} \setminus ([\![\tau_1^\epsilon]\!] \cup \ldots \cup [\![\tau_{n-1}^\epsilon]\!] \cup]\!]\tau_n^\epsilon, \infty[\![),$$

it follows inductively that, if X is predictable, then

$$[\![\tau_n^\epsilon]\!] \in \mathcal{P}.$$

Consequently $[\![\tau_n^\epsilon, \infty[\![= [\![\tau_n^\epsilon]\!] \cup]\!]\tau_n^\epsilon, \infty[\![\in \mathcal{P}$, which means that the τ_n^ϵ are predictable times. Now, the proof follows from the obvious inclusion

$$\left\{(t, \omega) : X_{t-}(\omega) \neq X_t(\omega)\right\} \subset \bigcup_{m,n} [\![\tau_n^{1/m}]\!].$$

Another proof may be obtained as a Corollary to the following.

Lemma. *Under Assumption* 2.1, *for each non-negative Borel function f on E^2 vanishing on the diagonal D of E^2 the equalities*

$$P_0^f = 0, \quad P_t^f = \sum_{0 < s \leq t} f(X_{s-}, X_s) \ \text{ if } t > 0,$$

define a $[0, \infty]$*-valued optional process which is predictable if X is predictable.*

Proof of the optional part of the Lemma is the same as in [5; 4.5]. Proof of the predictable part is similar. Suppose that X is predictable. Then, for each natural n and each Borel function f on E^2, the process

$$P_t^{f,n} = \sum_{k=1}^{n} f(X_{(k-1)t/n}, X_{kt/n}), \ t \geq 0,$$

is predictable. If f is continuous and such that

$$\operatorname{dist}(x, y) \leq \epsilon \Longrightarrow f(x, y) = 0$$

for some $\epsilon > 0$, then, similarly to [5; 4.3],

$$\lim_{n \to \infty} P^{f,n}(\omega) = P^f_t(\omega)$$

for each $(t, \omega) \in [0, \infty) \times \Omega$, so that P^f is predictable in this case. Finally, by a monotone class argument as in [5; 4.5 and 4.8], P^f is predictable for each Borel $f \geqslant 0$ vanishing on D.

Corollary. *Under Assumptions 2.1, let $B_1, B_2, \ldots$ be a sequence of disjoint Borel subsets of E^2 such that $\bigcup_m B_m = E^2 \setminus D$ and that* $\inf \left\{ \operatorname{dist}(x, y) : (x, y) \in B_m \right\} > 0$ *for each m. Write*

$$\tau_n^m = n - th \text{ debut of } \left\{ (t, \omega) : (X_{t-}(\omega), X_t(\omega)) \in B_m \right\} .$$

Then the τ_n^m are stopping times with disjoint graphs such that

$$\bigcup_{m,n} [\![\tau_n^m]\!] = \left\{ (t, \omega) : X_{t-}(\omega) \neq X_t(\omega) \right\} .$$

Moreover, if the process X is predictable, the the τ_n^m are predictable times.

Proof. We have $[\![\tau_n^m, \infty [\![= \left\{ (t, \omega) : P^f_t(\omega) \geqslant n \right\}$ with $f = \mathbb{1}_{B_m}$.

2.5. The necessity of condition (b) in Theorem 2.2 follows at once from 1.2.2.

2.6. Sufficiency of (a) and (b). In order to prove the sufficiency of (a) and (b) in Theorem 2.2 we shall use arguments from [2; IV, 88C] with some minor simplifications. Suppose that Assumptions 2.1 and conditions (a) and (b) are satisfied. Define the process X^- by

$$X_0^- = X_0, \; X_t^- = X_{t-} \text{ for } t > 0.$$

The processes X^- is $(\mathcal{F}_{t-})$-adapted and left-continuous. The latter implies that $X_t^-(\omega) = \lim_n X_t^n(\omega)$ for each $(t, \omega) \in [0, \infty) \times \Omega$, where $X_t^n = X^-_{[nt]/n}$. For each $B \in \mathcal{B}(E)$, we have $B_k = (X^-_{k/n})^{-1}(B) \in \mathcal{F}_{(k/n)-}$, $(X^n)^{-1}(B) = \bigcup_{k=0} ([k/n, \infty) \times B_k - [(k+1)/n, \infty) \times B_k) \in \mathcal{P}$. So, the process X^n are predictable, and so is X^-.

According to the condition (a), there is a sequence $\tau_1, \tau_2, \ldots$ of predictable times such that

$$\left\{ (t, \omega) : X_t^-(\omega) \neq X_t(\omega) \right\} \subset \bigcup_n [\![\tau_n]\!] .$$

By 1.3.2, the graphs $[\![\tau_1]\!], [\![\tau_2]\!], \ldots$ are predictable subsets of $[0, \infty) \times \Omega$ which implies that $C = [0, \infty) \times \Omega \setminus \bigcup_n [\![\tau_n]\!]$ is also predictable.

We have to prove that $X^{-1}(B) \in \mathcal{P}$ whenever $B \in \mathcal{B}(E)$. To this end, observe first of all that $X = X^-$ on C, so that

$$C \cap X^{-1}(B) = C \cap (X^-)^{-1}(B) \in \mathcal{P}$$

whenever $B \in \mathcal{B}(E)$. Since

$$X^{-1}(B) = (C \cap X^{-1}(B)) \cup \bigcup_n ([\![\tau_n]\!] \cap X^{-1}(B)),$$

it remains to ascertain that $[\![\tau]\!] \cap X^{-1}(B) \in \mathcal{B}$ for each predictable time τ and each $B \in \mathcal{B}(E)$. The latter is a consequence of condition (b), according to which $A = (X_\tau)^{-1}(B) \in \mathcal{F}_{\tau-}$ whenever τ is a predictable time and $B \in \mathcal{B}(E)$. Under the same conditions, by 1.3.3, τ_A is a predictable time and so, by 1.3.2, $[\![\tau]\!] \cap X^{-1}(B) = [\![\tau_A]\!] \in \mathcal{P}$.

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STRESZCZENIE

Praca zawiera niezależny od miary probabilistycznej dowód twierdzenia Dellacherie i Meyera [2], charakteryzującego procesy stochastyczne przewidywalne w klasie procesów stochastycznych, których wszystkie trajektorie są prawostronnie ciągłe i mają tylko skokowe nieciągłości.

РЕЗЮМЕ

Работа содержит независимыу от вероятностной меры доказательство теоремы Делгашери и Майера [2] характеризующей предсказуемые случайные процессы в классе всех резулярных случайных процессов.