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Instytut Matematyki, Fizyki i Chemii
Politechnika Lubelska
Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej

A. SZYNAL, J. SZYNAL

The Generalization of Jenkins Inequality

Uogólnienie nierówności Jenkinsa

Обобщение неравенства Дженкинса

1. Let S denote the familiar class of holomorphic and univalent functions f in the unit disk $|z|<1$ which have the form

$$f(z)=z+a_2z^2+\dots,\ |z|<1, \tag{1}$$

In [1] Jenkins obtained by means of his general coefficient theorem the sharp inequality for $f\in S$:

$$\operatorname{Re}\left[(a_3-a_2^2)+2x_0a_2\right]\leqslant 1+\frac{3}{2}x_0^2-x_0^2\log\frac{|x_0|}{2},\quad -2\leqslant x_0\leqslant 2, \tag{2}$$

which enables him to verify Bieberbach conjecture for the third coefficient, $|a_3|\leqslant 3$.

In this note we will prove the extension of Jenkins' inequality (2) in which the right hand side depends on the value of function in a fixed point of the unit disk.

As a corollary we get (2) as well another proof of the fact that $|a_3|\leqslant 3$ in S.

Our tool will be the Pederson–Schiffer–Lebedev inequalities [3], [2]. The proof of these inequalities in [3] is based on variational method. Hovever, Lebedev's proof [2] has an elementary character and follows directly from ordinary Grunsky inequalities and Bieberbach transformation for odd univalent functions from S. So we may claim that the inequality $|a_3|\leqslant 3$ is also the consequence of Grunsky inequalities.

2. Let a be arbitrary fixed number such that $0<|a|<1$ and $f(a)=d=Re^{i\phi}$, $f\in S$. Forming the function

$$F(z) = \sqrt{\frac{d - f(z)}{1 - \bar{d} f(z)}}, \quad f \in S, \tag{3}$$

we define the coefficients A_{mn}, a_{mn}, b_{mn} by the expansions ($|z|$ and $|\xi|$ are sufficiently small):

$$\log \frac{F(z) - F(\xi)}{[F(z) + F(\xi)] \cdot (z - \xi)} = \sum_{m,n=0}^{N} A_{mn} z^m \xi^n \tag{4}$$

$$\log \frac{\sqrt{(1 - z/a)(1 - \bar{a}\xi)} - \sqrt{(1 - \xi/a)(1 - \bar{a} z)}}{[\sqrt{(1 - z/a)(1 - \bar{a}\xi)} + \sqrt{(1 - \xi/a)(1 - \bar{a} z)}\,](z - \xi)} = \sum_{m,n=0}^{N} a_{mn} z^m \xi^n \tag{5}$$

$$\log \frac{\sqrt{(1 - \bar{a} z)(1 - a\bar{\xi})} + |a| \sqrt{(1 - z/a)(1 - \bar{\xi}/\bar{a})}}{\sqrt{(1 - \bar{a} z)(1 - a\bar{\xi})} - |a| \sqrt{(1 - z/a)(1 - \bar{\xi}/\bar{a})}} = \sum_{m,n=0}^{\infty} b_{mn} z^m \bar{\xi}^n \tag{6}$$

We have

Lemma 1. [2]. *Let $f \in S$ and the coefficients A_{mn}, a_{mn}, b_{mn} are given by* (4)–(6), *where F is defined in* (3). *Then for any complex numbers x_k, y_k, $k = 0, 1, \ldots, N$, $N = 0, 1, \ldots$, the following inequalities*

$$\Big| \sum_{m,n=0}^{N} (A_{mn} - a_{mn}) x_m y_n \Big|^2 \le \Big(\sum_{m,n=0}^{N} b_{mn} x_m \bar{x}_n \Big) \Big(\sum_{m,n=0}^{N} b_{mn} y_m \bar{y}_n \Big) \tag{7}$$

hold.

The expansions (4)–(6) and some formulae given in [2] allow us to find the coefficients A_{mn}, a_{mn}, b_{mn} which look as follows:

$$\begin{aligned}
&A_{mn} = A_{nm}, && a_{mn} = a_{nm}, && b_{mn} = \bar{b}_{nm}, \\
&A_{00} = \log\Big(\frac{-1}{4d}\Big), && a_{00} = \log\Big(-\frac{1 - |a|^2}{4a}\Big), && b_{00} = \log \frac{1 + |a|}{1 - |a|}, \\
&A_{01} = a_2 + \frac{1}{2d}, && a_{01} = \frac{1 + |a|^2}{2a}, && b_{01} = -\frac{a}{|a|}, \\
&A_{11} = (a_3 - a_2^2) + \frac{1}{8d^2}, && a_{11} = \frac{(1 - |a|^2)^2}{8a^2}, && b_{11} = \frac{1 + |a|^2}{2|a|}.
\end{aligned} \tag{8}$$

We will need the following result which belongs to Grunsky [e.g. 4]. Nevertheless this result follows from (7) as well if we put $x_0 = y_0 \neq 0$ and $x_k = y_k = 0$, $k = 1, 2, \ldots, N$.

Lemma 2. *If $f \in S$ then for fixed a, $0 < |a| < 1$, the region of variability of $\log f(a)/a$ is the closed disk*

$$\left| \log \frac{f(a)}{a} + \log(1 - |a|^2) \right| \leq \log \frac{1 + |a|}{1 - |a|} .$$

Corollary. *If $f \in S$ then for fixed a, $0 < |a| < 1$, the following sharp estimates*

$$\frac{|a|}{(1 + |a|)^2} \leq |f(a)| \leq \frac{|a|}{(1 - |a|)^2} \tag{9}$$

$$\left| \arg \frac{f(a)}{a} \right| \leq \log \frac{1 + |a|}{1 - |a|} \tag{10}$$

hold.

3. We have

Theorem 1. *If $f \in S$, then for any real x_0 the following inequality holds*

$$\operatorname{Re} [(a_3 - a_2^2) + 2x_0 a_2] \leq x_0^2 \log R \frac{(1+r)^2}{r} + x_0 [\frac{(1-r)^2}{r} \cos\theta - \frac{1}{R} \cos\phi] +$$

$$+ \frac{1}{8} [\frac{(1-r^2)^2}{r^2} \cos 2\theta - \frac{1}{R^2} \cos 2\phi] + \frac{1+r^2}{2r} , \tag{11}$$

where $a = re^{i\theta}$, $r \in (0, 1)$, $f(a) = \operatorname{Re}^{i\phi}$, θ and ϕ are real numbers.

Proof. Let us put $N = 1$, $x_0 = y_0 =$ real number and $x_1 = y_1 = 1$ in (7). We obtain the inequality:

$$|(A_{00} - a_{00}) x_0^2 + 2(A_{10} - a_{10}) x_0 + (A_{11} - a_{11})| \leq$$

$$\leq |b_{00}| x_0|^2 + 2 x_0 \operatorname{Re} b_{01} + b_{11}| . \tag{12}$$

Putting (8) into (12) we get

$$\Big| [(a_3 - a_2^2) + 2x_0 a_2] - \Big\{ x_0^2 \log f(a) \frac{(1 - |a|^2)}{a} +$$

$$+ x_0 (\frac{1 + |a|^2}{a} - \frac{1}{f(a)}) + \frac{1}{8} (\frac{(1 - |a|^2)^2}{a^2} - \frac{1}{f^2(a)}) \Big\} \Big| \leq \tag{13}$$

$$\leq \left| x_0^2 \log \frac{1 + |a|}{1 - |a|} - 2x_0 \operatorname{Re} \frac{a}{|a|} + \frac{1 + |a|^2}{2|a|} \right| .$$

The inequality (13) can be considered as the disk where lies the expression $[(a_3 - a_2^2) + 2x_0 a_2]$ for $f \in S$ and every real x_0.

Inequality (11) implies (13) if we take into account its geometric interpretation. In what follows we will use the denotations:

$$a \text{ is fixed number}, a = re^{i\theta}, r \in (0, 1); \tag{14}$$

$$A = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1}, \quad B = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1};$$

$$I(f) = \operatorname{Re}[(a_3 - a_2^2) + 2x_0 a_2], \ f \in S; \tag{15}$$

$$\Phi(R, \phi) = x_0^2 \log R - \frac{x_0}{R} \cos - \frac{1}{8R^2} \cos 2\phi, \quad f(a) = Re^{i\phi}; \tag{16}$$

$$\Psi(\theta, r, x_0) = x_0^2 \log \frac{(1+r)^2}{r} + x_0 \frac{(1-r)^2}{r} \cos \theta + \frac{(1-r^2)^2}{8R^2} \cos 2\theta + \frac{1+r^2}{2r}; \tag{17}$$

$$\cos \hat{\phi} = -2x_0 R, \quad \hat{\hat{\phi}} = \log \frac{1+r}{1-r}. \tag{18}$$

Theorem 3. *If $f \in S$ and a, $0 < |a| < 1$ is fixed number, then for any real x_0 the following inequalities hold:*

for $r \in (0, A)$ and $x_0 \geqslant 0$ (a_1)

$$I(f) - \Psi(0, r, x_0) \leqslant \Phi(R, \hat{\hat{\phi}}); \tag{19}$$

for $r \in (0, A)$ and $x_0 < 0$ (a_2)

$$I(f) - \Psi(0, r, x_0) \leqslant \begin{cases} \Phi(R, 0) & \text{if } -2x_0 R \geqslant 1 \\ \Phi(R, \hat{\phi}) & \text{if } \cos \hat{\hat{\phi}} \leqslant -2x_0 R \leqslant 1 \\ \Phi(R, \hat{\hat{\phi}}) & \text{if } \cos \hat{\hat{\phi}} \geqslant -2x_0 R; \end{cases} \tag{20}$$

for $r \in [A, B)$ and $x_0 \geqslant 0$ (b_1)

$$I(f) - \Psi(0, r, x_0) \leqslant \begin{cases} \Phi(R, \hat{\phi}) & \text{if } -2x_0 R \geqslant \cos \hat{\hat{\phi}} \\ \Phi(R, \hat{\hat{\phi}}) & \text{if } -2x_0 R \leqslant \cos \hat{\hat{\phi}}; \end{cases} \tag{21}$$

for $r \in [A, B)$ and $x_0 < 0$ (b_2)

$$I(f) - \Psi(0, r, x_0) \leqslant \begin{cases} \Phi(R, 0) & \text{if } -2x_0 R \geqslant 1 \\ \Phi(R, \hat{\phi}) & \text{if } -2x_0 R \leqslant 1; \end{cases} \tag{22}$$

for $r \in [B, 1)$ *and* $x_0 \geq 0$ (c_1)

$$I(f) - \Psi(0, r, x_0) \leq \begin{cases} \Phi(R, \hat{\phi}) & if\ -2x_0 R \geq -1 \\ \Phi(R, \pi) & if\ -2x_0 R \leq -1; \end{cases} \tag{23}$$

for $r \in [B, 1)$ *and* $x_0 < 0$ (c_2)

$$I(f) - \Psi(0, r, x_0) \leq \begin{cases} \Phi(R, \hat{\phi}) & if\ -2x_0 R \leq 1 \\ \Phi(R, 0) & if\ -2x_0 R \geq 1. \end{cases} \tag{24}$$

Proof. Under the denotations (15)–(17) inequality (11) takes the form

$$I(f) \leq \Phi(R, \phi) + \Psi(\theta, r, x_0).$$

Now, we are looking for maximal value of $\Phi(R, \phi)$ with respect to $\phi = \arg f(a)$. We can restrict ourselves to the case $\theta = 0$ because the class S is rotationaly invariant. From (10) we have $|\phi| = \hat{\hat{\phi}} = \log(1 + r)/(1 - r)$ and under this restriction we find $\max_\phi \Phi(R, \phi)$.

We have

$$\Phi'_\phi(R, \phi) = 0 \Longleftrightarrow \sin\phi(\cos\phi) + 2x_0 R) = 0.$$

So $\max_\phi \Phi(R, \phi)$ can be attained for $\phi = 0$ or $\phi = \pi$ or $\phi = \hat{\phi}$ or $\phi = \hat{\hat{\phi}}$ (if we forget about multiplicity of 2π).

Simple considerations imply:

$$\max_\phi \Phi(R, \phi) = \begin{cases} \Phi(R, 0) & \text{if } -2x_0 R \geq 1 \\ \Phi(R, \hat{\phi}) & \text{if } \cos\hat{\hat{\phi}} \leq -2x_0 R \leq 1 \\ \Phi(R, \hat{\hat{\phi}}) & \text{if } \cos\hat{\hat{\phi}} \geq -2x_0 R \end{cases}$$

in the case $r \in (0, B)$ and

$$\max_\phi \Phi(R, \phi) = \begin{cases} \Phi(R, 0) & \text{if } -2x_0 R \geq 1 \\ \Phi(R, \hat{\phi}) & \text{if } -1 \leq -2x_0 R \leq 1 \\ \Phi(R, \pi) & \text{if } -2x_0 R \leq -1, \end{cases}$$

in the case $r \in [B, 1)$.

If we take into consideration the cases for x_0 and $\cos\hat{\hat{\phi}} = \cos[\log(1 + r)/(1 - r)]$ where they are positive or negative then we get (19)–(24) which ends the proof.

From (23) and (24) one can get the following

Corollary 1. *Let* $f \in S$ *and* $x_0 \in R$. *If* $r \in [B, 1)$ *then*

$$I(f) \leqslant \begin{cases} x_0^2 + x_0^2 \log|f(a)| + \dfrac{1}{8|f(a)|^2} + \Psi(0, r, x_0) & \text{if } 2|x_0||f(a)| \leqslant 1 \\ x_0^2 \log|f(a)| + \dfrac{|x_0|}{|f(a)|} - \dfrac{1}{8|f(a)|^2} + \Psi(0, r, x_0) & \\ & \text{if } 2|x_0||f(a)| \geqslant 1. \end{cases} \tag{25}$$

In particular we have

Corollary 2. *Let $f \in S$ and $r \in [B, 1)$. Then for $x_0 \in R$ the following inequalities hold:*

$$I(f) \leqslant \begin{cases} 1 + \dfrac{3}{2} x_0^2 + x_0^2 \log \dfrac{1}{2|x_0|} + \Psi(0, r, x_0) & \text{if } \dfrac{(1-r)^2}{2r} \leqslant |x_0| \leqslant \dfrac{(1+r)^2}{2r} \\ x_0^2 \log \dfrac{r}{(1+r)^2} + |x_0| \dfrac{(1+r)^2}{r} - \dfrac{(1+r)^4}{8r^2} + \Psi(0, r, x_0) & \\ & \text{if } |x_0| \geqslant \dfrac{(1+r)^2}{2r}. \end{cases}$$

If we take into account that the function $\Psi(0, r, x_0)$ is decreasing w.r.t. $r \in (0, 1)$ then we get in the limit case $r \to 1$ the Jenkins inequality:

Corollary 3. *Let $f \in S$ and $x_0 \in R$. Then the following sharp inequalities hold:*

$$I(f) = \begin{cases} 1 + \dfrac{3}{2} x_0^2 - x_0^2 \log \dfrac{|x_0|}{2} & \text{if } |x_0| \leqslant 2 \\ 4|x_0| - 1 & \text{if } |x_0| \geqslant 2. \end{cases} \tag{26}$$

The extremal function in (26) is described in [1] for $|x_0| \leqslant 2$ and for $|x_0| \geqslant 2$ the extremal function is Koebe function.

Corollary 4. *Let $f \in S$ and $r \in [B, 1)$. If*

$$\left| \operatorname{Re} a_2 - \frac{(1-r)^2}{2r} \right| \leqslant \frac{1 + \log R\,[(1+r)^2/r]}{2R}$$

then

$$\operatorname{Re} a_3 \leqslant (\operatorname{Re} a_2)^2 - \frac{\left[\operatorname{Re} a_2 - \dfrac{(1-r^2)^2}{2r}\right]^2}{1 + \log R \dfrac{(1+r)^2}{r}} + \frac{1}{8R^2} + \frac{(1-r^2)^2}{8R^2} + \frac{1+r^2}{2r} \tag{27}$$

Proof. From the first line of (25) we have

$$\operatorname{Re} a_3 \leqslant \operatorname{Re} a_2^2 + x_0^2 \left[1 + \log R \frac{(1+r)^2}{r}\right] - 2x_0 \left[\operatorname{Re} a_2 - \frac{(1-r)^2}{2r}\right] + \tag{28}$$

$$+\frac{1}{8R^2}+\frac{(1-r^2)^2}{8R^2}+\frac{1+r^2}{2r} \quad \text{if } 2|x_0|R \leqslant 1 . \tag{28}$$

If we find minimum of the right hand side of (28) then we get (27).

Corollary 5. *If $f \in S$ and $r \in [B, 1)$, then*

$$\operatorname{Re} a_3 \leqslant \left[\frac{(1-r)^2}{2r}+\frac{1+\log R\,\frac{(1+r)^2}{r}}{2R}\right]^2-\frac{1+\log R\,\frac{(1+r)^2}{r}}{4R^2}+ \tag{29}$$

$$+\frac{1}{8R^2}+\frac{(1-r^2)^2}{8r^2}+\frac{1+r^2}{2r} .$$

Proof. It is sufficient to find the maximum of the right hand side of (27) w.r.t. $\operatorname{Re} a_2$.

Corollary 6. *If $f \in S$, then* $\operatorname{Re} a_3 \leqslant 3$ *and the* sign *of equality holds only for Koebe functions.*

Proof. By straightforward calculation we find that the right hand side of (29) as the function of $R \in \left[\frac{r}{(1+r)^2}, \frac{r}{(1-r)^2}\right]$ is decreasing, so its maximum is attained for $R=\frac{r}{(1+r)^2}$ which corresponds to the Koebe function. The value of this maximum is equal to

$$\frac{1}{8}\left[\frac{(1+r)^4}{r^2}+\frac{4(1+r^2)}{r}+\frac{5(1-r^2)^2}{r^2}+\frac{2(1-r)^4}{r^2}\right]$$

The above expression is decreasing in $(0, 1)$, which implies for $r \to 1$, that $\operatorname{Re} a_3 \leqslant 3$ for every function $f \in S$.

Corrolary 7. *If $f \in S$, then for every a, $0<|a|<1$*

$$\left|(a_3-a_2^2)-\left[\frac{(1-|a|^2)^2}{8a^2}-\frac{1}{8f^2(a)}\right]\right| \leqslant \frac{1+|a|^2}{2|a|} . \tag{30}$$

Inequality (30) *is sharp and the extremal function is Koebe function*

Proof. Put $x_0=y_0=0$, $x_1, y_1 \in C$, $x_1 \neq 0$, $y_1 \neq 0$, $x_k=0$, $y_k=0$ for $k \geqslant 2$ in (7).

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STRESZCZENIE

Praca zawiera zastosowanie nierówności Pedersona–Schiffera–Lebedeva [2], [3] do dowodu rozszerzonej wersji nierówności Jenkinsa dla funkcji klasy *S*.

Otrzymana nierówność uwzględnia wartość funkcji w ustalonym punkcie koła jednostkowego.

РЕЗЮМЕ

В работе поданы применения неравенства Педерсона–Шиффера–Лебедева [2], [3] для доказательства расширенного неравенства типа Дженкинса для класса *S*.

Полученное неравенство учитывает значение функций в фиксированной точке единичного круга.