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Analytic Functions with Univalent Derivatives

Funkcje analityczne z pochodnymi jednolistnymi

Аналитические функции с однолистными производными

1. Introduction. Let D be the unit disk $\{z : |z| < 1\}$ and let E be the family of functions $f(z) = z + a_2 z^2 + \dots$ that are analytic in D and such that the n^{th} derivative $f^{(n)}(z)$ is univalent in D, $n = 0, 1, 2, \dots, f^{(0)}(z) = f(z)$. This family was studied extensively by Shah and Trimble [3] – [8].

Let $\alpha = \sup \{|a_2| : f \in E\}$. Shah and Trimble showed that if $f \in E$ then f is entire and $|f(z)| \leqslant (e^{2\alpha|z|} - 1)/(2\alpha)$ for all z. When $f \in E$, it follows that $[f^{(n)}(z) - n!\, a_n]/((n+1)!\, a_{n+1}) \in E$ so that $|(n+2)a_{n+2}/(2a_{n+1})| \leqslant \alpha$. By induction, it therefore follows that $|a_n| \leqslant (2\alpha)^{n-1}/n!$. The above inequalities lead one to ask whether

$$f_\alpha(z) = \sum_{n=1}^{\infty} \frac{(2\alpha)^{n-1}}{n!} z^n = (e^{2\alpha z} - 1)/(2\alpha) \in E.$$

If so, $\alpha = \pi/2$ (the largest value of α that makes f univalent in the unit disk) and this led Shah and Trimble to conjecture that $\alpha = \pi/2$. However, M. Lachance [1] showed that $\alpha > 1.5910 > \pi/2 + .02$ by showing that $(e^{\pi z} - 1 + a(z + bz^2))/(\pi + a) \in E$ where $a = \pi e^{-\pi}/35$ and $b = 18$.

A somewhat simpler counterexample is obtained as follows. Let $h(z) = e^{\pi z} + ce^{-\pi z}$ where c is real. Suppose $|z| < 1$, $|w| < 1$ and $h(z) = h(w)$. Then $e^{\pi z} - e^{\pi w} = c(e^{\pi z} - e^{\pi w})e^{-\pi(z+w)}$ so that $c = e^{\pi(z+w)}$. Hence it follows that h is univalent if $-e^{-2\pi} \leqslant c \leqslant e^{-2\pi}$. Further, after normalizing, we conclude that

$$h_c(z) = \frac{1}{\pi} \sinh \pi z + \frac{1+c}{1-c} \cdot \frac{1}{\pi} (\cosh \pi z - 1) \in E \tag{1}$$

when $-e^{-2\pi} \leqslant c \leqslant e^{-2\pi}$. Setting $c = e^{-2\pi}$ yields the result

$$\max_{f\in E} |a_{2k}| \geqslant (\pi^{2k-1}/(2k)!) \coth \pi > (\pi^{2k-1}/(2k)!)(1.0037).$$

In view of these examples, it is somewhat surprising that if $f(z) = z + a_2 z^2 + \ldots$ has real coefficients with $a_{2k+1} > 0$, $k = 1, 2, \ldots$ and if $f \in E$ then $a_{2k+1} \leqslant \pi^{2k}/(2k+1)!$ with equality for each of the functions h_c given by (1), $-e^{-2\pi} \leqslant c \leqslant e^{-2\pi}$. Similar methods yield the result: If $f(z) = z + a_2 z^2 + \ldots \in E$ has positive coefficients and $f^{(n)}(D)$ is convex for $n = 0, 1, 2, \ldots$ then $a_k \leqslant 1/k!$, $k = 2, 3, \ldots$ with equality when $f(z) = e^z - 1$.

2. Proofs. It is convenient to write $f(z) = \sum_{k=1}^{\infty} \frac{1}{k!} b_k z^k$ so that $a_k = b_k/k!$ and to let E_p denote the collection of f in E so that $a_k > 0$ for each k. We require the following results of Pólya [2].

Lemma 1. *Let $f(z) = e^{-cz} f_1(z)$ where $c \geqslant 0$, and f_1 entire of genus zero having only positive zeros with γ the first zero of f. If*

$$\frac{-zf'(z)}{f(z)} = s_1 z + s_2 z^2 + \ldots$$

and

$$\frac{1}{f(z)} = t_0 + t_1 z + \ldots$$

then

$$\frac{t_0}{t_1} \leqslant \frac{t_1}{t_2} \leqslant \frac{t_2}{t_3} \leqslant \ldots \leqslant \gamma \leqslant \ldots \leqslant \frac{s_2}{s_3} \leqslant \frac{s_1}{s_2}.$$

The following lemma is the principal result required to obtain the coefficient bound for the odd coefficients of functions in the class E_p.

Lemma 2. *If $g(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} b_{2k+1} z^{2k+1}$ has positive coefficients and $g^{(2n)}(z)$ is typically real for $n = 0, 1, \ldots$ then $b_{2k+1} \leqslant \pi^{2k}$, $k = 1, 2, \ldots$ with equality if and only if $g(z) = (1/\pi) \sinh \pi z$.*

Proof. First note that $(1/\pi) \sinh \pi z$ satisfies the hypotheses of the lemma since $\operatorname{Im}(\sinh(\pi r e^{i\theta})) = \cosh(\pi r \cos\theta) \sin(\pi r \sin\theta)$ and the even derivatives of $\sinh z$ are again $\sinh z$. Further, it is sufficient to prove $b_3 \leqslant \pi^2$ to obtain $b_{2k+1} \leqslant \pi^{2k}$ by induction.

We have the system of inequalities

$$\operatorname{Im}(g^{(2n)}(i)) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} b_{2(k+n)+1} \geqslant 0. \tag{2}$$

Each of the series converges because g is typically real and hence $(1/6)\cdot b_3 \leqslant 3$. By induction, $b_{2k+1} \leqslant (18)^k$. We wish to eliminate all the coefficients except b_3 from the system (2). That is, we wish to choose $t_0 = 1$ and find $t_1, t_2, \ldots \geqslant 0$ (if possible) so that

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k t_\ell \frac{b_{2(k+\ell)+1}}{(2k+1)!} = 1 - (\frac{1}{6} - t_1)\, b_3 \geqslant 0 \tag{3}$$

The coefficient of b_{2k+1}, $k \geqslant 2$ on the left side of (3) is $t_k - (1/3!)\, t_{k-1} + \ldots + (-1)^k \dfrac{1}{(2k+1)!}$. Thus, in matrix form we want $AT = t_1 B + C$ where A is lower triangular with elements $A_{k,j} = A_{k-j+1,1} = (-1)^{k-j} \dfrac{1}{(2(k-j)+1)!}$ if $k \geqslant j$, $T = \begin{bmatrix} t_2 \\ t_3 \\ \vdots \end{bmatrix}$, B and C are column vectors with elements $B_j = (-1)^{j-1} \dfrac{1}{(2j+1)!}$ and $C_j = B_{j+1}$. If follows that A^{-1} is lower triangular with equal elements along diagonals. Write $(\beta_{j,k}) = A^{-1}$ and we then have $\beta_{j+\ell,\ell+1} = \beta_{j,1} = \beta_{j-1}$ where $\dfrac{\sqrt{z}}{\sin\sqrt{z}} = \sum_{j=0}^{\infty} \beta_j z^j$ (because $\dfrac{\sin\sqrt{z}}{\sqrt{z}} = \sum_{j=1}^{\infty} A_{j,1} z^{j-1}$). Therefore, the equation $T = A^{-1} B t_1 + A^{-1} C$ is equivalent to

$$\sum_{k=0}^{\infty} t_{k+2} z^k = \frac{\sqrt{z}}{\sin\sqrt{z}} \left[\frac{\sqrt{z} - \sin\sqrt{z}}{z^{2/3}}\, t_1 - \frac{\sin\sqrt{z} - \sqrt{z} + \frac{z^{3/2}}{6}}{z^{5/2}} \right] =$$

$$= \sum_{k=0}^{\infty} (\beta_{k+1} (t_1 - \frac{1}{6}) + \beta_{k+2})\, z^k$$

so that $t_{k+2} = \beta_{k+1} (t_1 - \frac{1}{6}) + \beta_{k+2}$. We require $t_{k+2} \geqslant 0$, $k = 0, 1, \ldots$ and hence

$$\frac{\beta_{k+2}}{\beta_{k+1}} \geqslant \frac{1}{6} - t_1 .$$

By Lemma 1 $\dfrac{\beta_{k+2}}{\beta_{k+1}}$ decreases and has limit $1/\pi^2$ as $k \to \infty$. Therefore $t_1 = \dfrac{1}{6} - \dfrac{1}{\pi^2}$ will imply $t_k > 0$ for $k = 2, 3, \ldots$. It remains to show that the series on the left side of (3) converges for then we have $0 \leqslant 1 - (\frac{1}{6} - t_1)\, b_3 = 1 - \frac{1}{\pi^2} b_3$ with equality if and only if $\operatorname{Im}(g^{(2n)}(i)) = 0$ for $n = 0, 1, 2, \ldots$. Thus, equality implies $b_{2k+1} = \pi^{2k}$ so $g(z) = \dfrac{1}{\pi} \sinh \pi z$.

As we have shown, the choice $t_0 = 1, t_1 = \frac{1}{6} - \frac{1}{\pi^2}, t_k = \beta_k - \frac{1}{\pi^2}\beta_{k-1}$ $k \geq 2$ will yield

$$0 \leq \sum_{\ell=0}^{N} t_\ell \operatorname{Im}(g^{(2\ell)}(i)) =$$

$$= 1 - \frac{1}{\pi^2} b_3 + \sum_{\ell=0}^{N} \sum_{k=N+1}^{\infty} (-1)^{k+\ell} t_\ell \frac{1}{(2(k-\ell)+1)!} b_{2k+1},$$

$$0 \leq 1 - \frac{1}{\pi^2} b_3 - \sum_{k=N+1}^{\infty} \sum_{\ell=N+1}^{k} (-1)^{k+\ell} t_\ell \frac{1}{(2(k-\ell)+1)!} b_{2k+1}. \tag{4}$$

Now suppose $\beta^2 = \sup b_3$ such that $g(z) = z + \frac{b_3}{3!} z^3 + \ldots$ has the property $g^{(2n)}(z)$ is typically real for each $n = 0, 1, \ldots$. Assume $\beta > \pi$ and assume g is chosen so that $b_3 = \beta^2$. Choose $e > 0$ so that $1/r\, g(rz) + ez^3 = g_e(z)$ is typically real whenever $r \leq \pi/\beta$. Then $g_e^{(2n)}(z)$ is typically real for each $n = 0, 1, \ldots$. Applying (4) to g_e with $r < \pi/\beta$ fixed yields

$$0 \leq 1 - \frac{1}{\pi^2}(b_3 r^2 + e) - \sum_{k=N+1}^{\infty} \sum_{\ell=N+1}^{k} (-1)^{k+\ell} \frac{1}{(2(k-\ell)+1)!} t_\ell b_{2k+1} r^{2k} \tag{5}$$

Since $b_{2k+1} \leq \beta^{2k}$, we have

$$\left| \sum_{k=N+1}^{\infty} \sum_{\ell=N+1}^{k} (-1)^{k+\ell} \frac{1}{(2(k-\ell)+1)!} t_\ell b_{2k+1} r^{2k} \right| \leq$$

$$\leq \sum_{\ell=N+1}^{\infty} \sum_{k=\ell}^{\infty} (\beta r)^{2k} t_\ell \frac{1}{(2(k-\ell)+1)!} = \sum_{\ell=N+1}^{\infty} (\beta r)^{2\ell} t_\ell \frac{\sinh \beta r}{\beta r}$$

Since $((\beta r)^{2\ell} t_\ell)^{1/\ell} = (\beta r)^2 t_\ell^{1/\ell} = (\beta r)^2 (\beta_\ell)^{1/\ell} (1 - \frac{1}{\pi^2} \frac{\beta_{\ell-1}}{\beta_\ell})^{1/\ell}$ while $(\beta r)^2 < \pi^2$, $1 - \frac{1}{\pi^2} \frac{\beta_{\ell-1}}{\beta_\ell} \to 0$ and $\limsup (\beta_\ell)^{1/\ell} = \frac{1}{\pi^2}$ we conclude $\sum_{\ell=0}^{\infty} (\beta r)^{2\ell} t_\ell$ is convergent so $\sum_{\ell=N+1}^{\infty} (\beta r)^{2\ell} t_\ell \to 0$ as $N \to \infty$. From (5), we now conclude $b_3 r^2 + e \leq \pi^2$ when $r < \pi/\beta$. Since $b_3 = \beta^2$, letting $r \to \pi/\beta$ we obtain $\pi^2 + e \leq \pi^2$ which is a contradiction. Hence we must have $\beta = \pi$ and the proof is complete.

We can now easily prove the following theorem.

Theorem 1. *If $f(z) = z + a_2 z^2 + \ldots$ has the property $f^{(2n)}(z)$ is typically real for $n = 0, 1, \ldots$ and $a_{2k+1} > 0$, $k = 1, 2, \ldots$, then $a_{2k+1} \leq \pi^{2k}/(2k+1)!$. If $a_{2k} > 0$ for $k = 1, 2, \ldots$ and $f^{(2n+1)}(z)$ is typically real for $n = 0, 1, \ldots$ then $a_{2k} \leq$*

$\leq 2a_2 \pi^{2(k-1)} / (2k)!$. *The inequalities are sharp with equality iff* $f(z) = h_c(z)$, $c = e^{-2\pi}$.

Proof. Apply Lemma 2 to $\frac{1}{2}(f(z) - f(-z))$ and $\frac{1}{4a_2}(f'(z) - f'(-z))$.

Note that we have not used the full strength of the hypotheses. In Lemma 2, we only require that $b_{2k+1} \leq \beta^{2k}$ for some β and that $\text{Im}(g^{(2n)}(i)) \geq 0$.

Now assume $f(z) = z + \frac{b_2}{2} z^2 + \frac{b_3}{3!} z^3 + \ldots$ is in the family E_p and that $f^{(k)}(|z| < 1)$ is convex for $k = 0, 1, \ldots$. We know that $zf^{(k)}(z)$ is starlike so $|f^{(k)}(z)| \geq |b_k| / 4$. Therefore, since $f^{(k)}(-r) = b_k - rb_{k+1} + \ldots > 0$ for small r, we know $f^k(-r) > 0$ when $0 \leq r \leq 1$. By convexity and the fact that f is entire (so $f^{(k)}(-1)$ exists) we have $\text{Re}(z f^{(n+1)}(z) / f^{(n)}(z) + 1) \geq 0$, $n = 1, 2, \ldots$, $|z| \leq 1$ and $-f^{(n+1)}(-1) / f^{(n)}(-1) + 1 \geq 0$. Multiplying by $f^{(n)}(-1)$ yields $f^{(n)}(-1) - f^{(n+1)}(-1) \geq 0$. That is,

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{b_{k+n}}{k!} + \frac{kb_{k+n}}{k!} \right) \geq 0 . \tag{6}$$

This is a system of inequalities and as before, we wish to find $t_\ell \geq 0$, $t_1 = 1$ so that

$$0 \leq 1 - (2 - t_2) b_2 = \sum_{\ell=1}^{\infty} t_\ell (f^{(\ell)}(-1) - f^{(\ell+1)}(-1)) .$$

Since the technique is identical to that used before, we omit the details. We only remark that in the application of Pólya's Theorem (Lemma 1), the function $\frac{\sin\sqrt{z}}{\sqrt{z}}$ of the previous proof is replaced by $e^{-z}(1 - z)$ Then $t_2 = 1$ and we obtain $0 \leq 1 - b_2$. The final result is the following theorem.

Theorem 2. *If* $f(z) = z + a_2 z^2 + \ldots \in E_p$ *has the property* $f^{(n)}(|z| < 1)$ *is convex for* $n = 0, 1, \ldots$ *then* $a_k \leq 1/k!$ *for* $k = 2, 3, \ldots$ *with equality if and only if* $f(z) = e^z - 1$.

Again, we have not used the full strength of the hypotheses. We have only used $f^{(n)}(-1) > 0$ for each $n = 1, 2, \ldots$ and $\text{Re}(zf^{(n+1)}(z) / f^{(n)}(z) + 1) > 0$ when $z = -1$ for each $n = 1, 2, \ldots$.

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STRESZCZENIE

Załóżmy, że E jest klasą funkcji

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

jednolistnych w kole $|z| < 1$ i takich, że $f^{(n)}(z)$, $n = 1, 2, \ldots$ są funkcjami jednolistnymi.
Dowodzi się, że jeżeli

$$f(z) = \overline{f(\bar{z})},\ a_{2k+1} > 0,\ k = 1, 2, \ldots, \quad \text{to} \quad a_{2k+1} \leq 2k/(2k+1)!.$$

Jeżeli $f \in E$, $a_k(f) > 0$ i $f(|z| < 1)$ jest obszarem wypukłym, to $a_k \leq 1/k!$.

РЕЗЮМЕ

Пусть E класс функций

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

однолистных в круге $|z| < 1$ и таких что $f^{(n)}(z)$, $n = 1, 2, \ldots$, однолистные функции. Доказывается, что если

$$f(z) = \overline{f(\bar{z})},\ a_{2k+1} > 0,\ k = 1, 2, 3, \ldots, \text{то}\ a_{2k+1} \leq 2k/(2k+1)!.$$

Если $f \in E$, $a_k(f) > 0$ и $f(|z| < 1)$ выпуклая область, то $a_k \leq 1/k!$.