ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. XXXVI/XXXVII, 16 SECTIO A 1982/1983

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Univalent Taylor Series with Integral Coefficients

Jednolistne szeregi Taylora o współczynnikach całkowitych

Однолистные ряды Тейлора с целыми коэффициентами

Let S denote the class of functions of the form $f(z) = z + \ldots$ that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. The Koebe functions $z/(1 \pm z)^2$ are extremal for many problems in S as are the functions $z/(1 \pm z)$ for the subfamily of S consisting of convex functions. The Taylor expansions for these four functions have integral coefficients. The question arises as to what other functions in S have only integral coefficients in their Taylor expansions. To find all such functions, we will make use of the following version of the classical

Area Theorem. *If $f(z) \in S$, then $\frac{1}{f(z)} = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$ satisfies the coefficient inequality*

$$\sum_{n=1}^{\infty} n |b_n|^2 \leq 1 . \tag{1}$$

We now prove our

Theorem. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ and a_n is an integer for every n, then $f(z)$ must have one of the forms*

1980 Mathematical subject classification. Primary 30C45.

* This work was completed while the second author was on sabbatical leave from the College of Charleston as a Visiting Scholar at the University of Michigan.

$$z, \frac{z}{1 \pm z}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{1 \pm z + z^2}.$$

Proof. Upon writing

$$\frac{1}{f(z)} = \frac{1}{z + \sum_{n=1}^{\infty} a_n z^n} = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n,$$

we note that $\{a_n\}$ and $\{b_n\}$ satisfy the relations

$$b_0 + a_2 = 0, \qquad b_1 + b_0 a_2 + a_3 = 0, \tag{2}$$

and, more generally,

$$b_n + b_{n-1} a_2 + b_{n-2} a_3 + \cdots + b_0 a_{n+1} + a_{n+2} = 0 \ (n \geq 2). \tag{3}$$

Since the $\{a_n\}$ are integers, it follows inductively from (3) that the $\{b_n\}$ are also integers. Hence, (1) implies that

$$|b_1| \leq 1 \tag{4}$$

and that $b_n = 0$ for $n = 2, 3, \ldots$. Thus, $\frac{1}{f(z)} = \frac{1}{z} + b_0 + b_1 z$ or, equivalently, $f(z)$ must have the form

$$f(z) = \frac{z}{1 + b_0 z + b_1 z^2}.$$

Now from (4) we know that the possible values for b_1 are $b_1 = 0, 1, -1$. Our result will follow from a consideration of these three cases.

Case (i): $b_1 = 0$. Then $f(z) = z / (1 + b_0 z)$ and since $f(z)$ is analytic in Δ, we must have $|b_0| \leq 1$, which yields the three possible values $b_0 = 0, 1, -1$. Thus, $f(z)$ has one of the forms

$$z, \ \frac{z}{1+z}, \quad \frac{z}{1-z}.$$

Case (ii): $b_1 = 1$. From the well-known bound $|a_2| \leq 2$ for functions in S, we see from (2) that the possible values for $b_0 = -a_2$ are $b_0 = 0, \pm 1, \pm 2$.

This generates the five functions

$$\frac{z}{1+z^2}, \quad \frac{z}{1+z+z^2}, \quad \frac{z}{1-z+z^2}, \quad \frac{z}{(1+z)^2}, \quad \frac{z}{(1-z)^2}.$$

each of which is univalent in Δ.

Case (iii): $b_1 = -1$. Since the denominator of $z/(1 + b_0 z - z^2)$ has its zeros at $z = (-b_0 \pm \sqrt{b_0^2 + 4})/2$, of the five possible values $0, \pm 1, \pm 2$ for b_0, only the case $b_0 = 0$ will produce a function anylytic in Δ. Consequently, our final case supplies us with the univalent functions $z/(1 - z^2)$.

Combining the three cases, we obtain the nine univalent functions whose Taylor series have integral coefficients. This completes the proof.

Note that the only functions in S with integral coefficients are rational (and starlike). This leads to the following questions:

(i) If $f(z)$ is in S and assumes rational values for z rational in Δ, must $f(z)$ be a rational function?

(ii) What can we say about rational functions in S?

Mitrinovič has some partial results [1] as do Reade and Todorov [2].

REFERENCES

[1] Mitrinovič, D. S., *On the univalence of rational functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 634 – 677, (1979), 221–227.

[2] Reade, M. O., Todorov, P. G., *The radii of starlikeness and convexity of order alpha of a rational function of Koebe type*, (submitted).

STRESZCZENIE

Znalezione zostały wszystkie funkcje klasy S o współczynnikach całkowitych.

РЕЗЮМЕ

Найдены все функции класса S с целыми коэффициентами.