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Growth of the Derivatives of Univalent and Bounded Functions

Wzrost pochodnych funkcji jednolistnych i ograniczonych

Рост производных ограниченных однолистных функций

1. Introduction. Let Σ denote the set of functions that are analytic and univalent in $\{z : 0 < |z| < 1\}$ and are normalized by

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \quad (0 < |z| < 1). \tag{1}$$

In [6] K. Löwner showed that if $f \in \Sigma$ then

$$|f'(z)| \leq \frac{1}{|z|^2 (1 - |z|^2)}, \quad (0 < |z| < 1). \tag{2}$$

Except for an additive constant there is a unique function in Σ for which equality in (2) holds at a point z_0. If $z_0 = r\ (0 < r < 1)$ the extremal functions are

$$f(z) = \frac{1}{z} + a_0 - \frac{(1 - r^2)\, z}{1 - rz} \tag{3}$$

Throughout this paper we let

$$M(r) = \max_{|z| = r} |f'(z)| \tag{4}$$

whenever f is analytic on $\{z : |z| = r\}$. Inequality (2) implies that $(1 - r)\, M(r)$ is uniformly bounded over Σ as $r \to 1$. Since the extremal functions for (2) vary with z_0 it is not clear whether is a function in Σ for which $\overline{\lim_{r\to 1}}\ (1-r)M(r) > 0$.

Our first theorem shows that this is not possible since

$$\overline{\lim_{r\to 1}}\ (1-r)M(r) = 0. \tag{5}$$

for each function in Σ. The proof is a consequence of the area theorem which asserts that

$$\sum_{n=1}^{\infty} n\,|a_n|^2 \leqslant 1 \tag{6}$$

whenever $f \in \Sigma$ [7, p. 210]. We also show that (5) is sharp in that there is no prescribed rate at which $(1-r)M(r)$ tends to zero for all functions in Σ. This is proved by an application of Ahlfor's distortion theorem to a suitable conformal mapping.

Similar results are obtained for the growth of the integral means. We shall let

$$I(f;\ r,\ p) = \frac{1}{2\pi}\int_0^{2\pi} |f'(z)|^p\, d\theta \tag{7}$$

where $z = re^{i\theta}$ and $p > 0$, whenever f is analytic on $\{z : |z| = r\}$. We show that $(1-r)^{p-1}\, I(f;\ r,\ p)$ is uniformly bounded over Σ as $r \to 1$ whenever $p \geqslant 2$. Also, if $p \geqslant 2$ and $f \in \Sigma$ then $(1-r)^{p-1}\, I(f;\ r,\ p) \to 0$ as $r \to 1$.

Let S denote the set of functions that are analytic and univalent in $\Delta = \{z : |z| < 1\}$ and are normalized by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (|z| < 1). \tag{8}$$

We show that the results quoted above have equivalent formulations for S which involve $g'(z)\,/\,g^2(z)$. We also use the arguments developed for Σ to obtain analogous results for the derivatives of bounded functions in S.

In the last section we obtain estimates on $M(r)$ and $I(f;\ r,\ p)$ for functions that are analytic and bounded in Δ. Examples are given which depend on infinite Blaschke products and gap series.

2. Meromorphic, univalent functions.

Theorem 1. *If* $f \in \Sigma$ *then* $\lim_{r\to 1}\ (1-r)M(r) = 0.$

Proof. Suppose that $f \in \Sigma$ and f has the Laurent expansion (1). If N is any positive integer then

$$r^2 M(r) \leqslant 1 + \sum_{n=1}^{N-1} n\,|a_n| + \sum_{n=N}^{\infty} n\,|a_n|\, r^{n+1}. \tag{9}$$

Cauchy's inequality implies that

$$\sum_{n=N}^{\infty} n|a_n| r^{n+1} \leqslant \left\{\sum_{n=N}^{\infty} n|a_n|^2\right\}^{1/2} \cdot \left\{\sum_{n=N}^{\infty} nr^{2(n+1)}\right\}^{1/2} \leqslant$$

$$\leqslant \left\{\sum_{n=N}^{\infty} n|a_n|^2\right\}^{1/2} \left\{\sum_{n=1}^{\infty} nr^{2(n-1)}\right\}^{1/2} =$$

$$= \frac{1}{1-r^2} \left\{\sum_{n=N}^{\infty} n|a_n|^2\right\}^{1/2}.$$

Using this inequality in (9) we obtain

$$r^2(1-r^2)M(r) \leqslant (1-r^2)\left[(1+\sum_{n=1}^{N-1} n|a_n|\right] + \left\{\sum_{n=N}^{\infty} n|a_n|^2\right\}^{1/2}. \tag{10}$$

If $\epsilon > 0$ then the convergence of the series in (6) implies that

$$\left\{\sum_{n=N}^{\infty} n|a_n|^2\right\}^{1/2} < \epsilon/2$$

for some integer N. With N so chosen, there is a number $\delta > 0$ so that

$$(1-r^2)[1+\sum_{n=1}^{N-1} n|a_n|] < \epsilon/2$$

whenever $1-\delta < r < 1$. Because of (10) this shows that $(1-r)M(r) \to 0$ as $r \to 1$.

Theorem 2. *Suppose that ϵ is a positive function defined on* $(0,1)$ *so that* $\epsilon(r) \to 0$ *as* $r \to 1$. *There is a function f in Σ for which*

$$\overline{\lim_{r\to 1}} \frac{(1-r)M(r)}{\epsilon(r)} = \infty. \tag{11}$$

Proof. Since $\epsilon(r) \to 0$ as $r \to 1$ there is an increasing sequence $\{\rho_n\}$ so that $\rho_1 > 0$, $\rho_n \to 1$ and $\epsilon(r) \leqslant 1/n$ whenever $\rho_n \leqslant r \leqslant \rho_{n+1}$. By approximating the step function defined by $\alpha(r) = 2/n$, if $\rho_n \leqslant r \leqslant \rho_{n+1}$, we obtain a function β which is differentiable on $[\rho_1, 1)$ and satisfies $\beta'(r) < 0$, $\beta(r) > \epsilon(r)$ and $\beta(r) \to 0$ as $r \to 1$. If $\gamma(r) = \beta(r) + \sqrt{1-r}$, then the function γ has the additional property that its graph has a vertical tangent at $(0, 0)$.

Let ϕ be an increasing differentiable function defined for $t \geqslant 0$ so that $\phi(t) \to +\infty$ as $t \to +\infty$. If $\omega(t) = \phi(t) + t^2 + t$ then ω has the additional properties that $\omega'(t) \geqslant 1 > 0$ and $\omega'(t) \to +\infty$ as $t \to +\infty$. Let $D = \{s : |\operatorname{Im} s| < \pi/2\}$. A simply connected domain E shall be defined in terms of ω. We require that $E \subset D$, and E contains and is symmetric with respect to the real-axis. Also, if the boundary of E is given by the curves $y = \lambda(x)$ and $y = -\lambda(x)$ then we require that for $x \geq t_0$,

$$\lambda(x) = \frac{\pi}{4\omega'(x)} . \tag{12}$$

Let ψ denote the analytic function that maps E one-to-one onto D so that $\psi(-\infty) = -\infty$, $\psi(0) = 0$ and $\psi(+\infty) = +\infty$. If t is real and $t > t_0$ then

$$\psi(t) > \psi(t_0) + \pi \int_{t_0}^{t} \frac{1}{2\lambda(x)}\, dx - 4\pi \tag{13}$$

whenever $\int_{t_0}^{t} 1 / (2\lambda(x))\ dx > 2$ [3, p. 136]. Equation (12) thereby implies that if t is sufficiently large, then $\psi(t) > \psi(t_0) + 2\,\omega(t) - 4\pi > \omega(t)$. Therefore,

$$\psi(t) > \phi(t) \tag{14}$$

for sufficiently large t. This asserts that E may be obtained so that on the positive real axis the mapping function ψ tends to ∞ as fast as we like. This is equivalent to having the inverse of ψ tend to ∞ as slowly as possible, in terms of a given monotone differentiable function.

Let $u = \phi(z)$ be the composite function given by $z \to s \to t \to u$ where $z \in \Delta$, $s = \log(1+z) / (1-z)$, $t = \log(1+u) / (1-u)$ and $s = \overline{\psi}(t)$. Then ϕ maps Δ one-to-one onto a subset of Δ and $\phi(z) \to 1$ as $z \to 1$. Since $z \to s$ and $t \to u$ are inverse mappings the previous argument implies that with γ given there is a domain E so that

$$\phi(r) < 1 - \gamma(r) \tag{15}$$

whenever $0 < r < 1$ and r is sufficiently close to 1.

We claim that there is an increasing sequence $\{r_n\}$ of positive numbers so that $r_n \to 1$ and

$$1 - \phi(r_n) \leq (1 - r_n)\,\phi'(r_n) \tag{16}$$

for $n = 1, 2, \ldots$ If no such sequence exists then there is number r_0 $(0 < r_0 < 1)$ so that $1 - \phi(r) \geq (1 - r)\,\phi'(r)$ whenever $r_0 \leq r < 1$. Integrating this inequality from r_0 to r we find that $\dfrac{1-\phi(r)}{1-r} \geq \dfrac{1-\phi(r_0)}{1-r_0}$ for $r_0 \leq r < 1$. This inequality is inconsistent with (15) and the fact that γ has a vertical tangent at (1, 1).

Using equations (16) and (15) and $\gamma(r) > \epsilon(r)$ we conclude that

$$(1 - r_n)\,\phi'(r_n) > \epsilon(r_n) \tag{17}$$

If $A = \phi'(0)$ then $A \neq 0$ and $f = (A/\phi) \in \Sigma$. Since $\phi(r_n) \to 1$ this implies that $(1 - r_n)\,|f'(r_n)| > (|A|/2)\,\epsilon(r_n)$ for sufficiently large n. If, in the initial argument, we replace ϵ by $\sqrt{\epsilon}$ this shows that there is a function f in Σ and a sequence $\{r_n\}$ so that $r_n \to 1$ and $\dfrac{(1-r_n)\,|f'(r_n)|}{\epsilon(r_n)} \to +\infty$. This proves (11).

The argument given in Theorem 2 depends only on a local property of f. Our example at $z = 1$ locally maps onto the exterior of a region with a suitable cusp. The next theorem indicates to what extent $|f'(z)|$ may tend to ∞ on an average. One assertion is uniform over Σ and the other holds for individual functions in Σ.

Theorem 3. *There is a positive constant C such that if $p \geq 2$ and $f \in \Sigma$ then*

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^p \, d\theta \leq \frac{C}{(1-r)^{p-1}}\,. \tag{18}$$

If $p \geq 2$ and $f \in \Sigma$ then

$$\lim_{r\to 1}\left\{(1-r)^{p-1}\,\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^p \, d\theta\right\} = 0\,. \tag{19}$$

Proof. Suppose that $f \in \Sigma$ and f has the expansion (1). Parseval's formula implies that

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta = 1 + \sum_{n=1}^{\infty} n^2\,|a_n|^2\, r^{2(n+1)}\,. \tag{20}$$

Since $r^{n+1}(1-r) \leq \max\limits_{0<r<1} r^{n+1}(1-r) = \left(\dfrac{n+1}{n+2}\right)^{n+1}\dfrac{1}{n+2}$ it follows that that

$$nr^{2(n+1)} \leq \frac{n}{n+1}\left(\frac{n+1}{n+2}\right)^{n+1}\frac{1}{1-r^2} \leq \frac{1}{2(1-r^2)}\,,$$ for $0 < r < 1$ and

$n = 1, 2, \ldots$ This inequality and (6) imply that $\sum\limits_{n=1}^{\infty} n^2\,|a_n|^2\, r^{2(n+1)} \leq \dfrac{1}{2(1-r^2)}$.

Because of (20) this proves (18) in the case $p = 2$ and with $C = 3/2$.

Now suppose that $p > 2$. We apply (2) and (18) in the case $p = 2$ to obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^p \, d\theta \leq \frac{1}{(1-r^2)^{p-2}}\,\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta \leq$$

$$\leq \frac{1}{(1-r^2)^{p-2}}\,\frac{3}{2(1-r^2)} = \frac{3}{2(1-r^2)^{p-1}} \leq \frac{3}{2(1-r)^{p-1}}\,.$$

We next prove (19) in the case $p = 2$. If N is any positive integer then from (20) we find that

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta \leq 1 + \sum_{n=1}^{N-1} n^2 |a_n|^2 + \frac{1}{2(1-r^2)} \sum_{n=N}^{\infty} n |a_n|^2 . \quad (21)$$

Suppose that $\epsilon > 0$. Since the series in (6) converges there is an integer N so that $\sum_{n=N}^{\infty} n|a_n|^2 < \epsilon$. Next δ is choosen so that $\delta > 0$ and $(1-r)\,[1 + \sum_{n=1}^{N-1} n^2 |a_n|^2\,] < \epsilon/2$ whenever $1 - \delta < r < 1$. Because of (21) this proves that $(1-r)\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta \to 0$ as $r \to 1$.

Now, suppose that $p > 2$ and $f \in \Sigma$. Inequality (2) and (19) in the case $p = 2$ imply that

$$(1-r)^{p-1} \frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^p \, d\theta \leq (1-r)^{p-1} \frac{1}{(1-r^2)^{p-2}} \cdot$$

$$\cdot \frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta \leq (1-r)\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^2 \, d\theta \to 0 \text{ as } r \to 1 .$$

Inequality (18) cannot be improved in the sense that if

$$A(p) = \sup_{0<r<1} \max_{f \in \Sigma} \left\{ (1-r)^{p-1} \frac{1}{2\pi}\int_0^{2\pi} \left| z^2 f'(z) \right|^p d\theta \right\} \quad (22)$$

then $A(p) > 0$ for $p \geq 2$. We need the following inequalities:

$$(a+b)^p \leq a^p + b^p \quad (a \geq 0,\ b \geq 0,\ 0 < p \leq 1), \quad (23)$$

$$(a+b)^p \leq 2^{p-1} \quad (a^p + b^p) \quad (a \geq 0,\ b \geq 0,\ p > 1). \quad (24)$$

These are proved in [2, p. 57] and combined assert that $(a + b)^p \leq C_p(a^p + b^p)$ where $C_p > 0$. If f is defined by equation (3), then to emphasize that f depends on r we write $f(z) = f_r(z)$. Since $z^2 f_r'(z) = -1 - [(1-r^2)\, z^2/(1-rz)^2]$ we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f_r'(z)|^p \, d\theta \geq C_p (1-r^2)^p r^{2p} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-rz|^{2p}} d\theta - 2\pi . \quad (25)$$

Where, as usual, $z = re^{i\theta}$. There are positive constants D_q so that if $q > 1$ and $z = Re^{i\theta}$ then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-z|^q} d\theta \geq \frac{D_q}{(1-R)^{q-1}} \quad (26)$$

[8, p. 262]. If (26) is used in (25) we see that

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f_r'(z)|^p \, d\theta \geq \frac{C_p D_{2p} r^{2p}}{(1-r^2)^{p-1}} - 1$$

whenever $p > ½$. This implies that $A(p) \geq (C_p D_{2p})/(2^{p-1})$ whenever $p > ½$. In particular, $A(p) > 0$ for $p \geq 2$.

The problem of the determining the best estimate on

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)|^p \, d\theta$$

where $f \in \Sigma$ and $p < 2$ seems to be difficult. The best known result in the case $p = 1$ is the assertion that

$$\frac{1}{2\pi}\int_0^{2\pi} |z^2 f'(z)| \, d\theta < \frac{A}{(1-r)^{1/2 - 1/300}} \qquad [1]$$

for a positive constant A.

3. **Analytic, univalent functions.** If $f \in \Sigma$ then there is a complex nuber c so that $f(z) \neq c$ for $0 < |z| < 1$. Thus the function $g = 1/(f - c) \in S$ and $-[(g'(z))/(g^2(z))] = f'(z)$. Conversely, if $g \in S$ then $f = (1/g) \in \Sigma$ and $f'(z) = -[(g'(z))/(g^2(z))]$. This implies that

$$\left\{-\frac{g'(z)}{g^2(z)} : g \in S\right\} = \left\{f'(z) : f \in \Sigma\right\} \qquad (27)$$

whenever $0 < |z| < 1$.

Because of (27) the results about Σ described in section 2 have equivalent formulations for S. For example, inequality (2) implies the sharp inequality

$$\left|\frac{z^2 g'(z)}{g^2(z)}\right| \leq \frac{1}{1 - |z|^2},$$

where $g \in S$ and $|z| < 1$, and inequality (18) implies that if $p \geq 2$ and $g \in S$ then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{z^2 g'(z)}{g^2(z)}\right|^p d\theta \leq \frac{C}{(1-r)^{p-1}}.$$

The arguments used to prove Theorems 1, 2 and 3 may be adapted to resolve similar problems for the derivatives of bounded functions in S. We shall outline how the arguments proceed and point out that the resultsdo not depend on the normalizations given for S.

Suppose that g is analytic in Δ and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \ (|z| < 1). \tag{28}$$

If g also satisfies $|g(z)| \leqslant M$, $(|z| < 1)$, then

$$|g'(z)| \leqslant \frac{M}{1 - |z|^2} \tag{29}$$

[5, p. 330]. If, in addition, g is univalent in Δ then as g maps Δ onto a set having area at most πM^2 we conclude that

$$\sum_{n=0}^{\infty} n \, |b_n|^2 \leqslant M^2 \tag{30}$$

The following theorem is a consequence of the convergence of the series in (30) and the proof is similar to the proof of Theorem 1.

Theorem 4. *If g is analytic, univalent and bounded in Δ then* $(1-r)\,M(r) \to 0$ *as* $r \to 1$.

Theorem 4 is sharp in the sense described in Theorem 2. This actually is shown in the proof of Theorem 2 where an extremal function g for this assertion is $g = \phi$, and say $M = 1$. The assertions of Theorem 3 also hold where f is replaced by g (and g is analytic, univalent and bounded). The argument depends on the inequalities (29) and (30). Inequality (18) is replaced by

$$\frac{1}{2\pi} \int_0^{2\pi} |z^2 g'(z)|^p \, d\theta \leqslant \frac{CM^2}{(1-r)^{p-1}} \tag{31}$$

where C is an absolute constant and $p \geqslant 2$.

4. Bounded, analytic functions. We now examine problems about the growth of $|g'(z)|$ where g is analytic and bounded in Δ (and not necessarily univalent) and for simplicity take the bound to be 1. Let $\mathcal{B}$ denote the set of functions g that are analytic in Δ and satisfy $|g(z)| \leqslant 1$ for $|z| < 1$.

Inequality (29) asserts that if $g \in \mathcal{B}$ then

$$|g'(z)| \leqslant \frac{1}{1 - |z|^2}, \ (|z| < 1). \tag{32}$$

Equality in (32) at $z = z_0$ $(|z_0| < 1)$ occurs only for the functions

$$g(z) = x \, \frac{z - z_0}{1 - \bar{z}_0 z} \tag{33}$$

where $|x| = 1$.

Since g in (33) depends on z_0 it isn't clear whether there is a function g in $\mathcal{B}$ for which

$$\overline{\lim_{r\to 1}}\ (1-r)\,M(r)>0\,. \tag{34}$$

We now provide an example where (34) holds. Suppose that

$$g(z)=\prod_{k=1}^{\infty}\frac{z-z_k}{1-\bar{z}_k z} \tag{35}$$

where $|z_k|<1$ and

$$\sum_{k=1}^{\infty}(1-|z_k|)<+\infty\,. \tag{36}$$

Condition (36) ensures that (35) converges in Δ uniformly on compact subsets [2, p. 19]. Since

$$|g'(z_n)|=\frac{1}{1-|z_n|^2}\prod_{k\neq n}^{\infty}\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right| \tag{37}$$

inequality (34) holds if there is a positive constant δ so that

$$\prod_{k\neq n}\left|\frac{z_n-z_n}{1-\bar{z}_k z_n}\right|\geqslant\delta\ \text{ for } n=1,2,\dots \tag{38}$$

Inequality (38) is the definition that $\{z_k\}$ is uniformly separated and a sufficient condition for this is

$$1-|z_{k+1}|\leqslant C\,(1-|z_k|)\ \text{ for } k=1,2,\dots \tag{39}$$

where $0<C<1$ [2, p. 155]. Thus, by letting $|z_k|\to 1$ geometrically we obtain our example. The example becomes even more interesting if $\{z_k\}$ is also choosen so that each point on $\partial\Delta$ is a point of accumulation of $\{z_k\}$.

The argument given to prove that (39) implies (38) shows that

$$\prod_{k\neq n}\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right|\geqslant\left[\prod_{n=1}^{\infty}\frac{1-C^n}{1+C^n}\right]^2. \tag{40}$$

Since the right-hand side of (40) tends 1 as $C\to 0$ we see that to each number A so that $0<A<1$, there is a function in $\mathcal{B}$ for which

$$\overline{\lim_{r\to 1}}\ (1-r^2)\,M(r)\geqslant A\,. \tag{41}$$

We raise the problem of whether there is a function in $\mathcal{B}$ for which

$$\overline{\lim_{r \to 1}} \; (1 - r^2)\, M(r) = 1 .$$

We next examine the growth of the integral means of the derivatives of functions in $\mathcal{B}$. The first theorem determines the exact upper bounds for these means when $0 < p \leq 2$. The following inequality is needed for that argument.

Lemma. *If m is a non-negative integer and*

$$\frac{m}{m+1} \leq r < \frac{m+1}{m+2} \tag{42}$$

then

$$nr^{n-1} \leq (m+1)\, r^m \text{ for } n = 1, 2, \ldots \tag{43}$$

Proof. We may assume that $r > 0$, and we let $n_0 = -1/\log r$. Since the functions $y = x - \log(1+x)$ and $y = \log(1+x) - [x/(1+x)]$ are increasing for $x > 0$,

$$k < \frac{1}{(\log(1 + \frac{1}{k}))} < k + 1 \text{ for } k = 1, 2, \ldots$$

Applying this inequality and (42) we conclude that

$$m < n_0 < m + 2 . \tag{44}$$

The function $\mu(n) = nr^{n-1}$ $(n > 0)$ is increasing for $0 < n < n_0$ and decreasing for $n > n_0$. If n varies over the posture integers then (44) implies that the maximum of μ occurs at m, $m+1$ or $m+2$. Now, $\mu(m) \leq \mu(m+1)$ as this is equivalent to $r \geq m / m+1$. Also, $\mu(m+2) < \mu(m+1)$ since this is equivalent to $r^2 < (m+1)/(m+2)$, which follows from $r < (m+1)/(m+2)$. This proves (43).

We also note that equality in (43) occurs only for $n = m+1$ when $m / m+1 < r < (m+1)/(m+2)$ and only for $n = m$ and $n = m+1$ when $r = m / m+2$.

Theorem 5. *If $g \in \mathcal{B}$ and $0 < p \leq 2$ then*

$$\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \leq (m+1)^p \, r^{mp} \tag{45}$$

where m is the greatest integer in $r/(1-r)$.

Proof. m is the integer for which $m \leq r/(1-r) < m+1$ and this inequality is the same as (42).

If $g \in \mathcal{B}$ and g has the representation (28) then

$$\sum_{n=0}^{\infty} |b_n|^2 \leq 1 \tag{46}$$

[2, p. 8]. The Lemma assert that if $A(r) = \sup \{ nr^{n-1} : n = 1, 2, ... \}$ then $A(r) = (m+1) r^m$. Thus,

$$\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^2 \, d\theta = \sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2(n-1)} \leq A^2(r) \sum_{n=1}^{\infty} |b_n|^2 \leq A^2(r).$$

This proves (45) in the case $p = 2$.

Now, suppose that $0 < p < 2$. Hölder's inequality completes the proof, as follows

$$\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^2 \, d\theta \right\}^{p/2} \leq \left\{ (m+1)^2 r^{2m} \right\}^{p/2} = (m+1)^p r^{pm}$$

The argument also shows that if $m / m+1 < r < (m+1)/(m+2)$ then equality in (45) holds only for the functions $g(z) = xz^{m+1}$ where $|x| = 1$. When $r = m / m+1$ equality occurs only for the functions $g(z) = xz^{m+1}$ and $g(z) = xz^m$ where $|x| = 1$.

The precise upper bounds given by (45) grow with the same order as the 'trivial' estimates given by (32). Namely, (32) implies that

$$\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \leq \frac{1}{(1-r^2)^p},$$

which is asymptotic to $1 / [2^p (1-r)^p]$ as $r \to 1$. On the other hand, when $r = m / m+1$ the right hand side of (45) becomes $[1 / (1-r)^p] \, r^{p[r/(1-r)]}$, which is asymptotic to $1 / [e^p (1-r)^p]$ as $r \to 1$.

Inequality (45) cannot hold for large values of p. This is a consequence of the fact that if g is analytic in Δ and $0 < r < 1$ then

$$\lim_{p \to \infty} \left(\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \right)^{1/p} = \max_{|z|=r} |g'(z)|. \tag{47}$$

If we let $g(z) = (z-r)/(1-rz)$ then the right hand side of (47) is $1/(1-r^2)$ and if $g(z) = xz^n$ ($|x| = 1$) then

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \right)^{1/p} = nr^{n-1}.$$

Our assertion follows from the inequality $\sup \left\{ nr^{n-1} : n = 1, 2, ... \right\} < 1/(1-r^2)$, which is not difficult to show.

Theorem 6. *If $g \in \mathcal{B}$ and $p > 0$ then*

$$\lim_{r \to 1} \left\{ (1-r)^p \frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^p \, d\theta \right\} = 0. \tag{48}$$

Proof. Using the notation in the proof of Theorem 5, we see that if N is a positive integer then

$$\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta \le \sum_{n=1}^{N-1} n^2\,|b_n|^2 + \sum_{n=N}^{\infty} n^2\,|b_n|^2\,r^{2(n-1)}. \tag{49}$$

If $n \geq 2$ then

$$\max_{0<r<1}\,(1-r)\,r^{n-1} = \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}$$

and thus

$$nr^{n-1} \le \frac{1}{1-r}\left(\frac{n-1}{n}\right)^{n-1} \le \frac{1}{2(1-r)} \quad \text{for } n = 2, 3, \ldots$$

Applying this inequality in (49) we conclude that

$$(1-r)^2\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta \le (1-r)^2\sum_{n=1}^{N-1} n^2\,|b_n|^2 + \tfrac{1}{4}\sum_{n=N}^{\infty}|b_n|^2. \tag{50}$$

Because the series (46) converges, by first choosing N large and then letting $r \to 1$ we conclude from (50) that (48) holds in the case $p = 2$.

If $0 < p < 2$ then Hölder's inequality implies that

$$(1-r)^p\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^p\,d\theta \le (1-r)^p\left\{\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta\right\}^{p/2} =$$

$$= \left\{(1-r)^2\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta\right\}^{p/2}.$$

Since (48) holds in the case $p = 2$ this proves (48) when $0 < p < 2$.

If $p > 2$ then inequality (32) implies that

$$(1-r)^p\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^p\,d\theta \le (1-r)^p\,\frac{1}{(1-r^2)^{p-2}}\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta =$$

$$= \frac{1}{(1+r^2)^{p-2}}\left\{(1-r)^2\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^2\,d\theta\right\} \to 0$$

as $r \to 1$.

Theorem 6 is precise in the following sense. If ϵ is a positive function on (0, 1) so that $\epsilon(r) \to 1$ as $r \to 1$ then there is a function g in $\mathcal{B}$ for which

$$\overline{\lim_{r\to 1}}\;\frac{(1-r)^p\,\frac{1}{2\pi}\int_0^{2\pi}|g'(z)|^p\,d\theta}{\epsilon(r)} = \infty \tag{51}$$

The proof of this fact is implicitly contained in an argument in [4, p. 219–222]. The appropriate function g, which is constructed in terms of ϵ, has the form

$$g(z) = \sum_{n=1}^{\infty} a_n z^{\nu n}, \quad (|z|<1) \tag{52}$$

where $\{a_n\}$ is a specific sequence of positive numbers for which $\sum_{n=1}^{\infty} a_n < 1$. The sequence $\{\nu_n\}$ of positive integers is increasing and selected to tend to ∞ sufficiently fast. The actual argument assumed that $0 < p \leqslant 1$ since it relied on (23). When $p > 1$ by appealing to (24) the same argument is possible. Thus, (51) holds for each $p > 0$.

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STRESZCZENIE

Badane są problemy wzrostu pochodnej i niektórych średnich całkowych w klasach funkcji jednolistnych.

РЕЗЮМЕ

Изучаются проблемы роста производной и некоторых интегральных средних в классах однолистных функций.