ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN–POLONIA
VOL. XXXVI/XXXVII, 10 SECTIO A 1982/1983

Instytut Ekonomii Politycznej i Planowania
Zakład Zastosowań Matematyki
Uniwersytet Marii Curie-Skłodowskiej

Z. LEWANDOWSKI

Some Remarks on Univalence Criteria

Pewne uwagi o kryteriach jednolistności

Некоторые замечания об условиях однолистности

1. Introduction. The purpose of this paper is to establish some theorems representing univalence criteria for regular functions. A fundamental role here is played by Theorem 2 as a preparatory theorem for other results. The proof of this theorem is based on Theorem 1 which is due to Pommerenke [3], (Corollary 3).

We begin with some notations: $\mathbb{C}$ – the compex plane; $\mathbb{R} = (-\infty, \infty)$; $E_r = \{ z \in \mathbb{C}: |z| < r \}$, $E_1 = E$; $\bar{A}$ – the closure of the set A; Ω – the class of functions ω which are regular in E and such that $|\omega(z)| \leq 1$ for $z \in E$; $K(S, R)$ – the open disc of the centre S and the radius R.

Theorem 1. *Let $r_0 \in (0, 1]$ and let $f(z, t) = a_1(t) z + ..., a_1(t) \neq 0$, be regular in E_{r_0} for each $t \in [0, \infty)$ and locally absolutely continuous in $[0, \infty)$, locally uniformly with respect to E_{r_0}. Suppose that for almost all $t \in [0, \infty)$ f satisfies the equation*

$$\frac{\partial f(z, t)}{\partial t} = z \frac{\partial f(z, t)}{\partial z} \, p(z, t), \; z \in E_{r_0}$$

where $p(z, t)$ is regular in E and Re *$p(z, t) > 0$ for $z \in E$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and if $\{ f(z, t) / a_1(t) \}$ forms a normal family in E_{r_0}, then for each $t \in [0, \infty)$ $f(z, t)$ has a regular and univalent extension to the whole disc E.*

2. The main results. Before the formulation of Theorem 2 we shall give a trivial but useful

Remark 1. Let $D \subset \mathbb{C}$ be a convex domain such that its boundary ∂D does not contain any rectilinear segment. Suppose that $A \in \bar{D} \setminus \{a\}$ and $w(\lambda) = \lambda A + (1 - \lambda) B \in \bar{D}$ for some $\lambda \in [0, 1]$ and $a \in \partial D$, where $A \neq B$. Then $w(\lambda_0) \in D$ for each $\lambda_0 \in (\lambda, 1)$.

We assume throughout the whole paper that a, s, κ are fixed numbers and such that $a > ½, s = \alpha + i\beta, \alpha > 0, \beta \in \mathbb{R}, \kappa = a/\alpha$.

We come now to the formulation and proof of

Theorem 2. *Let $f(z) = z + \ldots$ and $g(z)$ be regular in E with $f'(z) \neq 0$ and such that*

$$\left| \frac{zf'(z)}{f(z)g(z)} - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha}. \tag{1}$$

If

$$\left| |z|^{2\kappa} \frac{zf'(z)}{f(z)\,g(z)} + (1-|z|^{2\kappa}) \left[\frac{zf'(z)}{f(z)} + s\frac{zg'(z)}{g(z)} \right] - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha} \tag{2}$$

holds for $z \in E$ then f is univalent in E.

Proof. We consider the family of functions

$$f(z,t) = f(ze^{-st})\,[1 + (e^{2at} - 1)\, g(ze^{-st})]^s, \quad t \in [0, \infty). \tag{3}$$

It follows from (1) with $f'(z) \neq 0$ for $z \in E$ that $f(z)g(z) \neq 0$ for $z \in E$. Put $A(z,t) = 1 + (e^{2at} - 1)\,g(ze^{-st}), g(0) = c_0$. From (1) we obtain $\mathrm{Re}\dfrac{a\bar{s}c_0}{\alpha} > ½$ thus $c_0 \notin (-\infty, 0]$ and there exists a number $\rho > 0$ such that $|A(0,t)| = |1 + (e^{2at} - 1)c_0| > \rho$ for each $t \in [0, \infty)$. It follows that there exists a number $r_1 > 0$ such that $A(z,t) \neq 0$ for each $t \in [0, \infty)$ and $z \in E_{r_1}$. We have also $f_z'(0,t) = [e^{-t} + (e^{(2a-1)t} - e^{-t})\,c_0]^s \neq 0$ and because $a > ½$ $|f_z'(0,t)| \to \infty$ as $t \to \infty$; here $f_z'(0,t)$ denotes this continuous branch of the power for which $f_z'(0,t) = 1^s = 1$. It is not difficult to verify that $\{f(z,t)/f_z'(0,t)\}$ forms a normal family in E_{r_0} and that $f(z,t)$ is local absolutely continuous in $[0, \infty)$ uniformly with respect to E_{r_0} if $r_0 = ½ r_1$, say. This is guaranted, among other, by uniform continuity of $f_t'(z,t)$ on $[0,T] \times E_{r_0}$ where $T > 0$ is an arbitrarily chosen fixed number.

By simple calculation we obtain

$$\frac{f_t'(z,t)}{zf_z'(z,t)} = p(z,t) = -s + \frac{2as}{e^{-2at}\dfrac{\zeta f'(\zeta)}{f(\zeta)g(\zeta)} + (1 - e^{-2at})\left[\dfrac{\zeta f'(\zeta)}{f(\zeta)g(\zeta)} + s\dfrac{\zeta g'(\zeta)}{g(\zeta)}\right]}, \tag{4}$$

where $\zeta = ze^{-st}$. Let us denote by $d(z,t)$ the denominator of the right hand side of (4). It follows from the definition of $f(z)$ that $d(z,0) \neq 0$. Replacing z by ζ in (2) and putting $\lambda = |\zeta|^{2\kappa} = |z|^{2\kappa}e^{-2\kappa t\alpha}$, $\lambda_0 = e^{-2at}$, by definition of κ we obtain $\lambda\lambda_0^{-1} = |z|^{2\kappa} < 1$. Hence, for fixed $z \in E$ and $t \in [0, \infty)$, we see from (1) and Remark 1 that $d(z,t) \in K(as/\alpha, a|s|/\alpha)$, if $A(\zeta) \neq B(\zeta)$ or $d(z,t) \in \overline{K}(as/\alpha, a|s|/\alpha) \setminus \{0\}$, if $A(\zeta) = B(\zeta)$. Simultaneously we have $d(0, t) = e^{-2at}\,\bar{c}_0^{-1} + (1 - e^{-2at})$. (1) implies $\bar{c}_0^{-1} \in \overline{K}(as/\alpha, a|s|/\alpha) \setminus \{0\}$. Also, it is easy to verify that $1 \in K(as/\alpha, a|s|/\alpha)$. Then, by Remark 1, we obtain $d(0, t) \in K(as/\alpha, a|s|/\alpha)$ for $t \in (0, \infty)$. Thus, for each $t \in (0, \infty)$ and $z \in E$ we have

$$\left| d(z, t) - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha}. \tag{5}$$

Hence, $p(z, t)$ is regular in E for each fixed $t \in [0, \infty)$. From (4) we also obtain that inequalities $\operatorname{Re} p(z, t) > 0$ and (5) are equivalent. Thus $\operatorname{Re} p(z, t) > 0$ for each $t \in (0, \infty)$ and $z \in E$. In addition $\operatorname{Re} p(z, t) \geqslant 0$ for $z \in E$. By Theorem 1 f is univalent in E. The proof of Theorem 2 has been completed.

Let us observe that Theorem 2 can be stated in the following equivalent from

Theorem 3. *Let $f(z) = z + \ldots$ be regular in E with $f'(z) \neq 0$ there and put*

$$H_s(f, \omega, z) = (1 - s)\, \frac{zf'(z)}{f(z)} + s\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z\,\omega'(z)}{e^{i\gamma} - \omega(z)}\right),$$

for $z \in E$. If there exists $\omega \in \Omega$, $\omega \neq e^{i\gamma}$ and $\gamma = \arg s$ such that

$$\left| (1 - |z|^2)\,[\alpha/a\, H_s(f, \omega, z) + |s|\,\omega - s] - \frac{|s|\,\omega\,(1 - |z|^2)}{1 - |z|^{2\kappa}} \right| < \frac{|s|\,(1 - |z|^2)}{1 - |z|^{2\kappa}} \tag{6}$$

holds for some $a > ½$, $s = \alpha + i\beta$, $\alpha > 0$, $\kappa = a/\alpha$ then f is univalent in E.

To see this we choose $g(z) = (zf'(z)/f(z))\,[(as/\alpha) - (a|s|/\alpha)\,\omega(z)]^{-1}$ which satisfies (1) then a straighforward calculation shows that (2) and (6) are equivalent.

3. Corollaries and applications. If we assume $s = 1$ then by Theorem 3 we obtain

Corollary 1. *Let f be regular in E with $f'(0) \neq 0$. If there exist a number $a > ½$ and a function $\omega \in \Omega$, $\omega \neq 1$ such that*

$$\left| |z|^{2a}\omega(z) - (1 - |z|^{2a}) \left\{ \frac{1 - a}{a} + \frac{1}{a}\left[\frac{zf''(z)}{f'(z)} + \frac{z\omega'(z)}{1 - \omega(z)}\right] \right\} \right| < 1 \tag{7}$$

for $z \in E$ then f is univalent in E.

Assume now $a > ½$ and $\omega = (1 - a)\,a^{-1}$. Corollary 1 yields

Corollary 2. *Let f be regular in E with $f'(0) \neq 0$ and let*

$$\left| a - 1 - (1 - |z|^{2a})\, \frac{zf''(z)}{f'(z)} \right| < a, \tag{8}$$

then f is univalent in E.

The above statement had been obtained earlier by the author and J. Szynal at another occasion. Corollary 2 in turn implies the well known univalence criterion, c.f. [1].

We now give some applications of Thorem 2. To this end we will introduce some notations. Let H and G denote such classes of functions regular in E for which $f(0) = f'(0) - 1 = 0$ if $f \in H$ and $g(z) \neq 0$ for $z \in E$ if $g \in G$. Put $f_r(z) = (1/r) f(rz)$ for $f \in H$ and $g_r(z) = g(rz)$ for $g \in G$.

Let us observe now that inequalities (1) and (2) can be written in following forms

$$\left| e^{-i\gamma} \frac{zf'(z)}{f(z)\, g(z)} - \frac{a\,|s|}{\alpha} \right| < \frac{a\,|s|}{\alpha}, \tag{9}$$

$$\left| |z|^{2\kappa} e^{-i\gamma} \frac{zf'(z)}{f(z)\,g(z)} + (1 - |z|^{2\kappa})\, e^{-i\gamma} \left[\frac{zf'(z)}{f(z)} + s \frac{zg'(z)}{g(z)} \right] - \frac{a\,|s|}{\alpha} \right| < \frac{a\,|s|}{\alpha} \tag{10}$$

which are equivalent to (1) and (2) respectively.

The limit case $a \to \infty$ suggests the following

Corollary 3. *Let $f \in H$ and $g \in G$. Then f is univalent in E provided the conditions*

$$\operatorname{Re}\left[e^{-i\gamma} \frac{zf'(z)}{f(z)\,g(z)} \right] > 0 \text{ for } z \in E \text{ and} \tag{11}$$

$$\operatorname{Re}\left\{ e^{-i\gamma} \left[\frac{zf'(z)}{f(z)} + s \frac{zg'(z)}{g(z)} \right] \right\} > 0 \text{ for } z \in E \tag{12}$$

hold for some $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$, where $\gamma = \arg s \in (-\pi/2, \pi/2)$.

Proof. 1° We assume first that $\operatorname{Re}\left[e^{-i\gamma} \dfrac{zf'(z)}{f(z)\,g(z)} \right] > 0$, for $z \in E$. Let $\mathcal{P}$ denote the class of functions $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, $z \in E$ that satisfy the condition $\operatorname{Re} p(z) > 0$. It is well-known that if $S = (1 + r^2)/(1 - r^2)$, $R = 2r/(1 - r^2)$ then $|p(z) - S| \leq R$ for z in E_r, $0 < r < 1$. Put $A(z) = e^{-i\gamma} \dfrac{zf'(z)}{f(z)\,g(z)}$ and $B(z) = e^{-i\gamma} \left[\dfrac{zf'(z)}{f(z)} + s \dfrac{zg'(z)}{g(z)} \right]$.

It is easy to verify that $A(rz) = e^{-i\gamma} \dfrac{zf_r'(z)}{f_r(z)\,g_r(z)}$ and $B(rz) = e^{-i\gamma} \left[\dfrac{zf_r'(z)}{f_r(z)} + s \dfrac{zg_r'(z)}{g_r(z)} \right]$.

In the considered case, by definitions and hypotheses of Corollary 3, $A(z)$ and $B(z)$ have positive real parts in E. Hence by an easy calculation and the mentioned property of $p \in \mathcal{P}$ we conclude that for a fixed $z \in E$, $A(rz)$ and $B(rz)$ lie in the closed discs $\overline{K}(S_1, R_1)$ and $\overline{K}(S_2, R_2)$ respectively where $\operatorname{Re}(S_1 - R_1) = [(1-r)/(1+r)]\,|a_0|^{-1} \cos(\gamma + \arg a_0)$ and $\operatorname{Re}(S_2 - R_2) = [(1-r)/(1+r)] \cos\gamma$, $a_0 = g(0)$. In addition in view of the assumption $\operatorname{Re}[e^{-i\gamma} zf'(z)/(f(z)g(z))] > 0$ for $z \in E$ there is $-\pi/2 < \arg a_0 + \gamma < \pi/2$. Also $-\pi/2 < \gamma < \pi/2$ by the assumption of the corollary. Hence $\operatorname{Re}(S_1 - R_1) > 0$ and $\operatorname{Re}(S_2 - R_2) > 0$. Thus we obtain that there exists $a > ½$ and such that $[K(S_1, R_1) \cup$

$\cup K(S_2, R_2)] \subset K(a|s|/\alpha, a|s|/\alpha)$ for a fixed $r \in (0, 1)$. Hence $A(rz)$ and $B(rz)$ are contained in $K(a|s|/\alpha, a|s|/\alpha)$. Simultaneously for each fixed $z \in E$ $\left| |z|^{2\kappa} A(rz) + (1 - |z|^{2\kappa}) B(rz) \right| \in K(a|s|/\alpha, a|s|/\alpha)$. Thus $f_r(z)$ and $g_r(z)$ satisfy (9) and (10) and $f_r(z)$ is univalent in E by Theorem 2. Hence f as the limit of f_r for $r \to 1$ is univalent in E.

2°. Suppose now that $\operatorname{Re}\left\{e^{-i\gamma} zf'(z)/[f(z)g(z)]\right\} = 0$ at some points of E. From the minimum principle of harmonic functions we obtain $\operatorname{Re}\left\{e^{-i\gamma} zf'(z)/[f(z)g(z)]\right\} \equiv 0$ for $z \in E$. Thus $\left\{e^{-i\gamma} zf'(z)/[f(z)g(z)]\right\} = ci$ for some $c \in \mathbb{R}$. Hence $ci g(z) = e^{-i\gamma} \dfrac{zf'(z)}{f(z)}$ and consequently $\dfrac{zg'(z)}{g(z)} = 1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}$. Thus $\operatorname{Re}\left\{e^{-i\gamma}[(1 - s) zf'(z)/f(z) + s(1 + zf''(z)/f'(z))]\right\} > 0$ for $z \in E$ by (12). We can write the last inequality in the following equivalent form

$$\operatorname{Re}\left\{\left[e^{-i\gamma} - |s|\right]\frac{zf'(z)}{f(z)} + |s|\left[\frac{zf''(z)}{f'(z)}\right]\right\} > 0 \text{ for } z \in E, \tag{13}$$

which is a known sufficient condition for univalence of f [2]. The proof of Corollary 3 is complete. From Corollary 3 we will deduce here two results first of which is equivalent to Corollary 3.

Corollary 4. *Let $f \in H$, $p \in \mathcal{P}$ and let a, γ, ϕ_0 be fixed numbers such that $\alpha \geqslant 0$, $\gamma \in (-\pi/2, \pi/2)$ and $(\gamma + \phi_0)$ $(-\pi/2, \pi/2)$. Then f is univalent in E provided*

$$\operatorname{Re}\left\{(e^{-i\gamma} - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z) + i\operatorname{tg}(\gamma + \phi_0)}\right]\right\} > 0 \tag{14}$$

for $z \in E$.

Proof. Let us put in (11) $e^{-i\gamma} zf'(z)/[f(z)g(z)] = p_0(z)$, $\operatorname{Re} p_0(z) \geqslant 0$ for $z \in E$. If $\operatorname{Re} p_0(z) = 0$ at some points of E then from case 2° of the proof of Corollary 3 we obtain (13) and consequently f is univalent in E. Thus we may assume that $\operatorname{Re} p_0(z) > 0$ for $z \in E$. By the choice of $p_0(z)$ we obtain $p_0(0) = |c_0|^{-1} e^{-i(\gamma + \phi_0)}$, where $c_0 = g(0)$, $\phi_0 = \arg c_0$. In addition $(\gamma + \phi_0) \in (-\pi/2, \pi/2)$ because $\operatorname{Re} p_0(0) > 0$. Hence $p_0(z) = p(z)|c_0|^{-1}\cos(\gamma + \phi_0) + i|c_0|^{-1}\sin(\gamma + \phi_0)$ where $p \in \mathcal{P}$. Moreover

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \frac{zp_0'(z)}{p_0(z)}, \quad \frac{zp_0'(z)}{p_0(z)} = \frac{zp'(z)}{p(z) + i\operatorname{tg}(\gamma + \phi_0)}.$$

Combining these equalities with (12) we obtain

$$\operatorname{Re}\left\{(e^{-i\gamma} - |s|)\frac{zf'(z)}{f(z)} + |s|\left[1 + \frac{zf''(z)}{f'(z)}\right] - |s|\frac{zp'(z)}{p(z) + i\operatorname{tg}(\gamma + \phi_0)}\right\} > 0 \tag{15}$$

for $|s| > 0$, $\gamma = \arg s$. Thus f is univalent by Corollary 3. We may take $|s| = \alpha > 0$. If now $\alpha = 0$ in (13) then f is a spiral-like univalent function. The proof of Corollary 4 has been completed.

Remark 2. Let B denote the class of functions $f \in H$ which satisfy Corollary 4. It is not difficult to verify that B is the well-known class of Bazilewich (c f.p.ex. [3], p. 166). To see this one ought to solve the differential equation

$$(e^{-i\gamma}-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left[\frac{zf''(z)}{f'(z)}-\frac{zp'(z)}{p(z)+i\operatorname{tg}(\gamma+\phi_0)}\right]=p_1(z),$$

where $p_1(0)=e^{-i\gamma}$ and $\operatorname{Re} p_1(z)>0$ for $z\in E$.

Corollary 5. *Let* $p(z)=1+p_1z+\ldots\in\mathcal{P}$ *with* $p_1\neq 0$. *Then* p *is univalent in* E *provided for some* $\alpha>0$ *the inequality*

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)-1}+\alpha\left[1+\frac{zp''(z)}{p'(z)}-\frac{zp'(z)}{p(z)}-\frac{zp'(z)}{p(z)-1}\right]\right\}>0$$

holds in E.

Corollary 5 follows from Corollary 3 by taking

$$f(z)=\frac{p(z)-1}{p_1},\quad g(z)=\frac{zp'(z)}{p(z)\,[p(z)-1]},\ \text{and } \gamma=0.$$

We come now to concluding remarks. The consideration contained in the proof of Theorem 2, from the very beginning to relation (4) is similar to that in [5], [excluding some modification as in nature].

A similar consideration can be also found in an earlier paper of Ruscheweyh [4]. But we inserted in the paper the mentioned fragment of the proof of Theorem 2 for the considerations to be complete.

The paper [5] contains a fundamental result which is stated as Theorem 1 and yields a sufficient condition for univalence of a regular function. That theorem can be applied, as it follows from its proof, if $a\geqslant\alpha$ only, while Theorem 2 can be applied without this restriction. We showed here that Theorem 2 is more general than Theorem 1 from [5] also in the case $0<\alpha\leqslant a$.

To this end we will now cite Theorem 1 from [5] as Theorem 4 almost literally.

Theorem 4. *Let* $f(z)=z+\ldots$ *and* $P(z)=1+c_1z+\ldots$ *be analytic in* E, $f(z)f'(z)/z$ *and* $P(z)$ *be different from zero for* z *in* E *and* $s=\alpha+i\beta, a>½, 0<\alpha\leqslant a$,

$$M=(\alpha/a)\,|s|+((\alpha/a)-1)\,|s+cP(z)|, \tag{16}$$

where $c\neq 0$ *is a complex number such that*

$$|s+cP(z)|\leqslant(\alpha\,|s|)\,/\,(2a-\alpha)\,. \tag{17}$$

Then $f(z)$ *is univalent in* E *if*

$$\left|\frac{\alpha}{a}(1-|z|^2)\left[(1-s)\frac{zf'(z)}{f(z)}+s\left(1+\frac{zf''(z)}{f'(z)}-\frac{zP'(z)}{P(z)}\right)\right]-\left[s+c|s|^2P(z)\right]\right| \leqslant \qquad\qquad \leqslant M \tag{18}$$

for z in E.

Note that relations (17) and (18) can be written as a single inequality which is equivalent to (17) and (18). Essentialy, (17) implies that there exists $\omega \subset \Omega$ such that $s + cP(z) = |s|\,\omega(z)$ with $|\omega(z)| \leqslant \alpha/(2a - \alpha) \leqslant 1$ for $z \in E$, where $\alpha/(2a-1) = 1$ iff $\alpha = a$. Combining this with (16) and (18) we get by suitable transformations, the following inequality

$$\left|(1-|z|^2)\left[\frac{\alpha}{a}H_s(z,f,\omega)+|s|\,\omega(z)-s\right]-|s|\,\omega\right| \leqslant \frac{\alpha}{a}|s|+\left(\frac{\alpha}{a}-1\right)|s|\,|\omega(z)|, \tag{19}$$

where $H_s(z,f,\omega_0) = (1-s)\dfrac{zf'(z)}{f(z)} + s\left(1+\dfrac{zf''(z)}{f'(z)} - \dfrac{z\,\omega'(z)}{e^{i\gamma}-\omega(z)}\right)$., $\gamma = \arg s$.

The relation (6) in Theorem 3 can be written in an equivalent form

$$\left|(1-|z|^2)\left[\frac{\alpha}{a}H_s(z,f,\omega)+|s|\omega(z)-s\right]-|s|\omega(z)\phi\left(|z|,\frac{a}{\alpha}\right)\right| \leqslant |s|\phi\left(|z|,\frac{a}{\alpha}\right), \tag{20}$$

where $\phi\left(|z|,\dfrac{a}{\alpha}\right) = \dfrac{1-|z|^2}{1-|z|^{2(a/\alpha)}}$, $z \in E$.

Note, that $\phi(x;\lambda) = (1-x^2)/(1-x^{2\kappa})$ decreases in $[0,1]$ from 1 to $1/\lambda$ for each fixed $\lambda > 1$ and $\phi(x;1) \equiv 1$. Note, that we assume $\phi(1) = \lim\limits_{x\to 1^-}\phi(x,\lambda) = 1/\lambda$ and

$0 < \alpha \leqslant a$ by the hypothese. Let now $z \in E$ be a fixed point. It can be verified by using the mentioned property of ϕ that $\overline{K}\,(|s|\,\omega(z)\ \phi(|z|,a/\alpha);\ |s|\,\phi(|z|,a/\alpha))$ contains the circle $\overline{K}\,(|s|\,\omega(z)\,\alpha/a;\ |s|\,\alpha/a)$. Thus every function f which satisfies the inequality

$$\left|(1-|z|^2)\left[\frac{\alpha}{a}H_s(z,f,\omega)+|s|\omega(z)-s\right]-|s|\omega(z)\frac{\alpha}{a}\right| \leqslant |s|\frac{\alpha}{a} \tag{21}$$

for a fixed $\omega \in \Omega$ satisfies also inequality (20). This is so because $z \in E$ was arbitrarily chosen. Hence we obtain the following

Corollary 6. *If f satisfies the assumptions of Theorem 3 and it is subjected to* (21) *then f is univalent in E.*

Note that a reasoning similar to above implies that every function satisfying (19) satisfies also (21). Hence Theorem 4 is a special case of Theorem 3.

Remark 3. We can also prove an analogy of Theorem 2 with an application to a function g of the form $g(\zeta) = \zeta + b_0 + b_1\zeta^{-1} + \dots$ which is regular in $E^0 \setminus \{\infty\}$ where $E^0 = \{\zeta \in \overline{\mathbb{C}} : |\zeta| > 1\}$. The following theorem is true.

Theorem 5. *Suppose that* $g(\zeta) = \zeta + b_0 + b_1\zeta^{-1} + \dots$ *and* $h(\zeta) = 1 + c_2\zeta^{-2} + \dots$ *are* *regular in $E^0 \setminus \{\infty\}$ and E^0, respectively, with $g'(\zeta) \neq 0$ for $\zeta \in E^0$. Let for some numbers $s = \alpha + i\beta, \alpha > 0, \beta \in \mathbb{R}, \frac{1}{2} < a \leqslant \alpha$ the inequality*

$$\left| \frac{\zeta g'(\zeta)}{g(\zeta)\, h(\zeta)} - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha}$$

hold in E^0. If the inequality

$$\left| |\zeta|^{2\kappa} \frac{\zeta g'(\zeta)}{g(\zeta)\, h(\zeta)} + (1 - |\zeta|^{2\kappa}) \left[\frac{\zeta g'(\zeta)}{h(\zeta)} \right] - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha}$$

holds for $\zeta \in E^0$ and $\kappa = a/\alpha$ then f is univalent in E^0.

Detailed considerations are contained in another paper which is to be published in Annales Polonici Mathematici (1985).

REFERENCES

[1] Becker, J., *Löwnersche Differetialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. reine angew. Math., 255 (1972), 23–43.

[2] Eenigenburg, P. J., Miller, S. S., Mocanu, P. T., Reade, M. O.; *On a subclass of Bazile ič functions*, Proc. Amer. Math. Soc., 45 (1974), 88–92.

[3] Pommerenke, Ch., *Über die Subordination analytischer Funktionen*, J. reine angew. Math., 218 (1965), 159–173).

[4] Ruscheweyh, S., *An extension of Becker's univalence condition*, Math. Ann., 220 (1976), 285–290.

[5] Sing, V., Chichra. Pran Nath., *An extension of Becker's criterion of univalence*, Journal of the Indian Math. Soc., 41 (1977), 353–361.

STRESZCZENIE

Praca zawiera następujący wynik podstawowy

Twierdzenie 2. Niech $a > \frac{1}{2}$, $s = \alpha + i\beta$, $\alpha > 0, \beta \in \mathbb{R}$, $\kappa = a/\alpha$ będą ustalonymi liczbami. Załóżmy, że $f(z) = z + \ldots$ i $g(z)$ są funkcjami regularnymi w $E = \{z: |z| < 1\}$ takimi, że $f'(z) \neq 0$ i $g(z) \neq 0$ w E oraz, że zachodzi nierówność

$$\left| \frac{zf'(z)}{f(z)\, g(z)} - \frac{a|s|}{\alpha} \right| < \frac{a|s|}{\alpha}.$$

Jeśli ponadto mamy

$$\left| |z|^{2K} \frac{zf'(z)}{f(z)\, g(z)} + (1 - |z|^{2K}) \left[\frac{zf'(z)}{f(z)} + s \frac{zg'(z)}{g(z)} \right] - \frac{as}{\alpha} \right| < \frac{a|s|}{\alpha},$$

to f jest funkcją jednolistną w E.

Praca zawiera pewne wnioski i zastosowania jak również analogon bez dowodu twierdzenia 2 dla funkcji $g(\zeta) = \zeta + b_0 + b_1 \zeta^{-1} + \ldots$ regularnej w $E^0 = \{\zeta: |\zeta| > 1\}$.

РЕЗЮМЕ

Работа содержит следующий результат

Теорема 2. Пусть $a > ½$, $s = \alpha + i\beta$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\kappa = a / \alpha$ фиксированные числа. Предположим, что функции $f(z) = z + ...$ и $g(z)$ регуллярны в $E = \{z: |z| < 1\}$, $f'(z) \neq 0$, $g(z) \neq 0$ для $z \in E$ и такие, что имеет место неравенство

$$\left| \frac{zf'(z)}{f(z)\, g(z)} - \frac{a\, s}{\alpha} \right| < \frac{a\,|s|}{\alpha} .$$

Если кроме того имеем

$$\left| |z|^{2\kappa} \frac{zf'(z)}{f(z)\, g(z)} + (1 - |z|^{2\kappa}) \left[\frac{zf'(z)}{f(z)} + s \frac{zg'(z)}{g(z)} \right] - \frac{a\, s}{\alpha} \right| < \frac{a\,|s|}{\alpha} ,$$

то f однолистна в E.

Работа содержит некоторые следствия и применения а также аналог теоремы 2 (без доказательства) для функции $g(\zeta) = \zeta + b_0 + b_1 \zeta^{-1} ...$ резулярной в $E^*_\zeta = \{\zeta: |\zeta| > 1\}$.