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University of Michigan
An Arbor, Michigan, USA
Indiana University
Bloomington, Indiana, USA

P. DUREN, G. SCHOBER

Nonvanishing Univalent Functions. II *

O funkcjach jednolistnych różnych od zera. II

Об отличных от нуля однолистных функциях. II

This paper is a continuation of our previous work [1] on the class S_0 of nonvanishing univalent functions. The class S_0 consists of all functions f analytic and univalent in the unit disc $\mathbb{D}$, with $f(z) \neq 0$ in $\mathbb{D}$ and $f(0) = 1$. In [1] we used a variational method to study linear extremal problems in S_0. For the special problem of minimum real part we obtained detailed information about the extremal functions. The present paper is directed primarily to the minimum real part problem. We point out and partially correct an error in [1], leaving the main results intact. We also reexamine a conjecture made in [1] and introduce a conformal mapping technique which leads to a more accurate calculation of a certain bifurcation point, where the character of the extremal function appears to change. Finally, we generalize the minimum real part problem and obtain further information on the region of values of functions in S_0 at a specified point.

1. Review of previous results. For fixed $\zeta \in \mathbb{D}$ we consider the problem of minimizing Re $\{f(\zeta)\}$ among all functions $f \in S_0$. There is no loss of generality in assuming $0 < \zeta < 1$. Let $k_0(z) = [(1+z)/(1-z)]^2$ be the 'Koebe function' for S_0, which maps $\mathbb{D}$ onto the complement of the negative real axix. We showed in [1] that Re $\{f(\zeta)\} \geq k_0(-\zeta)$ for all $\zeta \leq 3 - \sqrt{8} = 0.171...$, and that no rotation of k_0 minimizes Re $\{f(\zeta)\}$ for $\zeta > 2 - \sqrt{3} = 0.267...$

An extremal function f must map $\mathbb{D}$ onto the complement of an analytic arc Γ which extends from 0 to ∞ and satisfies

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$$\frac{B(B-1)}{w(w-1)(w-B)}\,dw^2>0,\ B=f(\zeta). \tag{1}$$

The omitted arc Γ is monotonic with respect to the family of ellipses with foci at 0 and 1, and makes an angle of less than $\pi/4$ with each orthogonal hyperbola (with foci at 0 and 1) it crosses. In view of the differential equation (1), this $\pi/4$ – property is expressed by the condition

$$\operatorname{Re}\left\{\frac{w-B}{B(B-1)}\right\}>0,\ w\in\Gamma,\ w\neq 0, \tag{2}$$

which places Γ in a certain half-plane bounded by a line through B.

In [1] we deduced from (1) that

$$J(B)=\int_C\sqrt{\frac{B(B-1)}{w(w-1)(w-B)}}\,dw>0$$

for a suitable determination of the square-root, where C is the image under f of the linear segment from 0 to ζ. This arc C extends from 1 to B and is a trajectory of the quadratic differential (1). It is clear that C does not meet Γ. Under the normalizing assumption that $\operatorname{Im}\{B\}\geq 0$, we claimed to prove in [1] (Theorem 8) that Γ lies in the lower half-plane. This allowed us to conclude that on the Riemann sphere punctured at 0, 1, B, and ∞, the arc C is homotopic to the linear segment from 1 to B. Parametrizing this segment by

$w=1+(B-1)t,\ 0\leq t\leq 1$,

we could then express J in the form

$$J(B)=\int_0^1\left\{\frac{B(1-B)}{1-t(1-B)}\right\}^{1/2}\frac{dt}{\sqrt{t(1-t)}}. \tag{3}$$

Unfortunatelly, the proof of Theorem 8 in [1] contains an error, leaving the full truth of that theorem in doubt. If $\operatorname{Re}\{B\}\geq 0$, however, it follows immediately from (2) that the point 1 lies in the half-plane forbidden to Γ, so that Γ cannot meet the linear segment from 1 to B. This is enough to ensure that J has the equivalent form (3) when $\operatorname{Re}\{B\}\geq 0$.

Furthermore, Brown [3] has observed that there is another trajectory of the quadratic differential (1) joining 1 to B. Thus Γ cannot wind around the point 1, and so we are always free to deform the arc C to the linear segment from 1 to B. Consequently, J has the form (3) in any case.

We found in [1] that if B is in the first quadrant it must lie in the quarter-disc defined by $\operatorname{Re}\{B\}\geq 0$, $\operatorname{Im}\{B\}\geq 0$, and $|B|\leq 1$. Numerical calculations, as reported in [1], show that the locus of points B in the first quadrant satisfying $\operatorname{Im}\{J(B)\}=0$ consists of the real segment from 0 to 1 and a curve which leaves the real axis at a point $Q\approx 0.36$

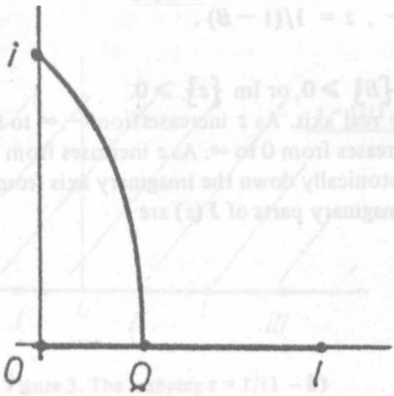

Figure 1. Solution set of Im $\{J(B)\} = 0$ in first quadrant

and goes with monotonic real and imaginary parts to the point i. (See Figure 1.) We proved by direct estimation ([1], § 8) that $J(i) > 0$ and Im $\{J(B)\} \neq 0$ elsewhere on the positive imaginary axis, confirming that $B = i$ actually arises from the extremal problem. Setting $B = i$ in the differential equation (1), one easily verifies that the (unique) trajectory emanating from the origin is simply the radial half-line $w = -(1+i)t$, $t > 0$. This leads to the sharp inequality ([1], Theorem 7) Re $\{f(z)\} > 0$ for all $f \in S_0$ and $|z| < \frac{1}{2}\sqrt{2-\sqrt{2}} = 0.382\ldots$.

Because $k_0(-\frac{1}{4}) = 0.36$, we conjecture in [1] that $Q = 0.36$ and that the sharp inequality

$$\text{Re } \{f(z)\} > k_0(-|z|), \ f \in S_0 ,$$

holds for $|z| < \frac{1}{4}$ but not for $|z| > \frac{1}{4}$. In the next section, however, we present a new approach which yields the more accurate value $Q = 0.36019\ldots$, corresponding to $z = -0.24987\ldots$. This imposes a slight modification on the conjecture and removes all hope of an elementary solution.

In § 3 we establish some properties of the omitted arc Γ which serve as partial substitutes for those asserted in Theorem 8 of [1]. In § 4 we generalize the minimum real part problem and the phenomenon of the isolated radial-slit solution. This gives new information on the region of values of $f(\zeta)$ at a fixed point $\zeta \in \mathbb{D}$ as f ranges over the class S_0. Although Hamilton [2] has described this region in terms of the elliptic modular function, its specific properties are not easily deduced.

2. Calculation of the bifurcation point. We now turn to the more accurate calculation of the bifurcation point Q of the curve Im $\{J(B)\} = 0$, where J is defined as the integral (3). It is convenient to introduce the function

$$\mathrm{J}(z) = \frac{i}{\sqrt{B}} \, J(B) = \int_0^1 \frac{dt}{\sqrt{(t-z)\,t\,(1-t)}} \ , \quad z = 1/(1-B) \, .$$

Without loss of generality we may assume $\mathrm{Im}\,\{B\} > 0$, or $\mathrm{Im}\,\{z\} > 0$.

We study first the behavior of $\mathrm{J}(z)$ on the real axis. As z increases from $-\infty$ to 0 on the negative real axis, $\mathrm{J}(z)$ is positive and increases from 0 to ∞. As z increases from 1 to ∞ on the positive real axis, $\mathrm{J}(z)$ comes monotonically down the imaginary axis from $i\infty$ to 0. On the segment $0 < z < 1$, the real and imaginary parts of $\mathrm{J}(z)$ are

$$\mathrm{J}_1(z) = \mathrm{Re}\,\{\mathrm{J}(z)\} = \int_z^1 \frac{dt}{\sqrt{(t-z)\,t\,(1-t)}}$$

and

$$\mathrm{J}_2(z) = \mathrm{Im}\,\{\mathrm{J}(z)\} = \int_0^z \frac{dt}{\sqrt{(z-t)\,t\,(1-t)}} \, .$$

As z increases from 0 to 1, the function $\mathrm{J}_1(z)$ is positive and decreases from ∞ to π, while $\mathrm{J}_2(z)$ increases from π to ∞. Because of the identity $\mathrm{J}_1(z) = \mathrm{J}_2(1-z)$, the function J maps the segment $(0, 1)$ onto a curve which is symmetric about the ray inclined at 45°. Since J is univalent on the real axis, it maps the upper half-plane univalently onto a region in the first quadrant as shown in Figure 2.

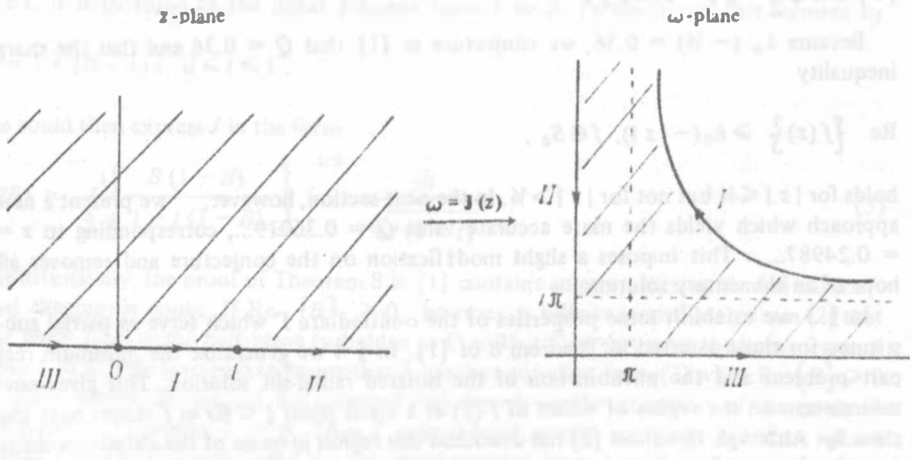

Figure 2. The mapping $\omega = \mathrm{J}(z)$

Next observe that the function $z = 1/(1-B)$ maps the upper half-plane onto itself as shown in Figure 3.

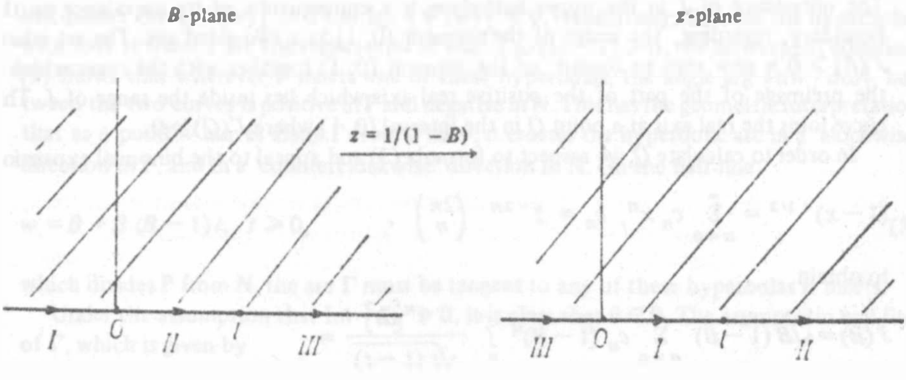

Figure 3. The mapping $z = 1/(1-B)$

Now consider the mapping

$$w = \mathrm{J}(B) = -i\sqrt{B}\ \mathrm{J}(1/(1-B)).$$

As B increases along the positive real axis from 1 to ∞, it is clear that $\mathrm{J}(B)$ falls monotically down the negative imaginary axis from 0 to $-i\infty$. As B increases along the negative real axis from $-\infty$ to 0, it is easily seen that $\operatorname{Re}\{\mathrm{J}(B)\} = \sqrt{-B}\ \mathrm{J}_1(1/(1-B))$ decreases from ∞ to 0, while $\operatorname{Im}\{\mathrm{J}(B)\} = \sqrt{-B}\ \mathrm{J}_2(1/(1-B)) = \sqrt{(1-z)/z}\ \mathrm{J}_2(z)$ is positive. A closer inspection reveals that $\operatorname{Im}\{\mathrm{J}(B)\}$ goes from ∞ to 0 as B increases from $-\infty$ to 0.

For $0 < B < 1$, we already know that $\mathrm{J}(B) > 0$, and we will show that $\mathrm{J}(B)$ increases from 0 to a maximum value and then decreases to 0. Putting all of this information together, we conclude that J maps the upper half-plane univalently onto a domain in the right half-plane as shown in Figure 4.

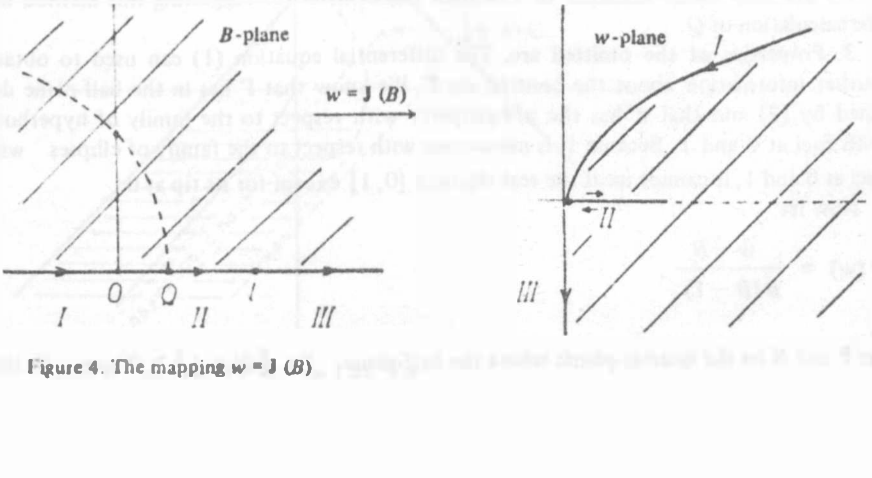

Figure 4. The mapping $w = \mathrm{J}(B)$

The univalence of J in the upper half-plane is a consequence of its univalence on the boundary, regarding the image of the segment (0, 1) as a two-sided slit. The set where $J(B) > 0$ is now seen to consist of the segment (0, 1) together with the curve which is the preimage of the part of the positive real axis which lies inside the range of J. This curve joins the real axis at a point Q in the interval (0, 1) where $J'(Q) = 0$.

In order to calculate Q, we respect to formula (3) and appeal to the binomial expansion

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} c_n x^n, \quad c_n = 2^{-2n} \binom{2n}{n},$$

to obtain

$$J(B) = \sqrt{B(1-B)} \sum_{n=0}^{\infty} c_n (1-B)^n \int_0^1 \frac{t^n \, dt}{\sqrt{t(1-t)}} =$$

$$= \pi \sqrt{B(1-B)} \sum_{n=0}^{\infty} c_n^2 (1-B)^n, \quad 0 < B < 1.$$

An easy calculation now gives $\frac{2}{\pi} \sqrt{B(1-B)}\, J'(B) = \sum_{n=1}^{\infty} d_n (1-B)^n - 1$, where $d_n = 2n c_{n-1}^2 - (2n+1) c_n^2 > 0$. Thus $\sqrt{B(1-B)}\, J'(B)$ is decreasing and so $J'(B)$ vanishes only once in the interval (0, 1). The point Q is therefore determined by the condition $J'(Q) = 0$, or

$$\sum_{n=1}^{\infty} d_n (1-Q)^n = 1, \quad 0 < Q < 1. \tag{4}$$

Observe also that $J'(B) > 0$ for $0 < B < Q$, while $J'(B) < 0$ for $Q < B < 1$. This shows that J has the monotonic property described above.

A numerical calculation based on the formula (4) gives $Q = 0.36019\ldots$ and $\zeta = 0.24987\ldots$ as the number for which $k_0(-\zeta) = Q$.

We are very much indebted to Friedrich Huckemann for suggesting this method for the calculation of Q.

3. Properties of the omitted arc. The differential equation (1) can used to obtain further information about the omitted arc Γ. We know that Γ lies in the half-plane defined by (2) and that it has the $\pi/4$-property with respect to the family of hyperbolas with foci at 0 and 1. Because Γ is monotonic with respect to the family of ellipses with foci at 0 and 1, it cannot meet the real segment [0, 1] except for its tip at 0.

Now let

$$\Psi(w) = \frac{w - B}{B(B-1)}.$$

Let **P** and **N** be the quarter-planes where the half-plane $\operatorname{Re}\{\Psi(w)\} > 0$ intersects the half-planes $\operatorname{Im}\{\Psi(w)\} > 0$ and $\operatorname{Im}\{\Psi(w)\} < 0$, respectively. Because the hyperbolas with foci at 0 and 1 are the trajectories of $d\omega^2 / \omega(\omega - 1) > 0$, the differential equation (1) shows that wherever Γ meets one of these hyperbolas, the angle $\arg\{dw / d\omega\}$ between the two curves is positive in **P** and negative in **N**. This has the geometric interpretation that as a point w moves along Γ from 0 to ∞, it crosses the hyperbolic arc in a 'clockwise' direction in **P**, and in a 'counterclockwise' direction in **N**. On the half-line

$$w = B + B(B-1)t, \quad t \geq 0, \tag{5}$$

which divides **P** from **N**, the arc Γ must be tangent to any of these hyperbolas it meets.

Under the assumption that $\operatorname{Im}\{B\} > 0$, it is clear that $0 \in \mathbf{P}$. The asymptotic half-line of Γ, which is given by

$$w = \tfrac{1}{3}(B+1) + B(B-1)t, \quad t \geq 0,$$

is parallel to (5) and also lies eventually in **P**. The latter statement follows from the inequality $\operatorname{Im}\left\{\dfrac{1-2B}{B(B-1)}\right\} > 0$, or $\operatorname{Im}\left\{B^2 - (1 + 2|B|^2)B\right\} < 0$, or $2x < 1 + 2(x^2 + y^2)$, where $B = x + iy$ and $y > 0$.

We now assert that the arc Γ is entirely confined to the quarter-plane **P**. Indeed, if it ever enters **N**, it must violate its elliptic monotonicity as it crosses the boundary (5) in order to approach its asymptotic half-line in **P**. Thus Γ moves in a clockwise direction with respect to the confocal hyperbolic arcs.

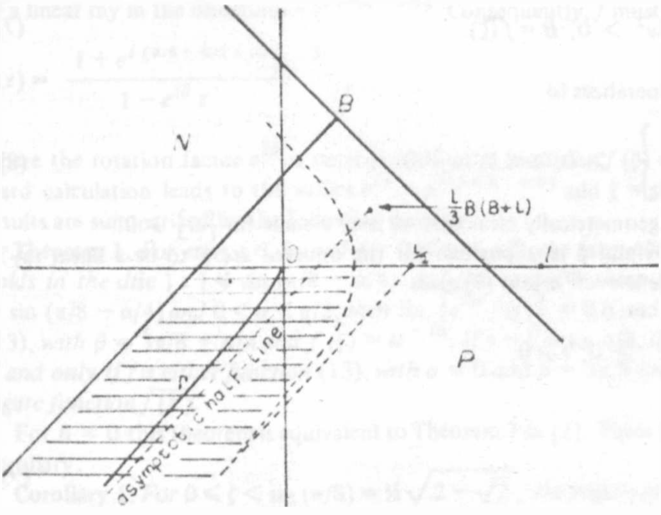

Figure 5. Location of Γ for $\operatorname{Re}\{B\} > 0$

If $\operatorname{Re}\{B\} \geq 0$, then $\operatorname{Re}\{\Psi(1)\} \leq 0$ and so the point 1 lies in the half-plane forbidden to Γ. This prohibits Γ from winding around the segment $[0, 1]$ and actually confines it to the part of the lower half-plane between the half-line (5) and the hyperbolic arc with asymptote $w = \frac{1}{2} + B(B-1)t$, $t \geq 0$, (See shaded region in Figure 5.) Indeed, if Γ ever crosses this hyperbolic arc, it must later recross in the opposite (counter-clockwise) direction in order to approach its asymptotic half-line.

We claimed in [1] (Theorem 8) that $\arg\{w - B\}$ is monotonic on Γ and that Γ is in a sector in the lower half-plane, but the proof was incorrect. Recently, however, Brown [3] has shown that $\arg w$ is monotonic on Γ and that is in a sector in the lower half-plane.

4. A more general problem. For a fixed angle α in the interval $-\pi < \alpha \leq \pi$, we now consider the more general extremal problem

$$\min_{f \in S_0} \operatorname{Re}\left\{e^{i\alpha} f(\zeta)\right\}, \quad 0 < \zeta < 1. \tag{6}$$

Because S_0 is preserved under conjugation, we may suppose without loss of generality that $0 \leq \alpha \leq \pi$. The minimum is attained for some function $f \in S_0$, and not for $f(z) \equiv 1$. For instance,

$$\operatorname{Re}\left\{e^{i\alpha}(1 - e^{-i\alpha}\zeta)\right\} = \cos\alpha - \zeta < 1. \tag{6 a}$$

The choice $\alpha = \pi$ corresponds to the maximum real part problem, equivalent to the maximum modulus problem and solved by the Koebe function k_0.

An application of the variational method shows as in [1] that an extremal function f for the problem (6) must map $\mathbb{D}$ onto the complement of an analytic arc Γ extending from 0 to ∞ and satisfying

$$\frac{e^{i\alpha} B(B-1)}{w(w-1)(w-B)}\, dw^2 > 0, \quad B = f(\zeta). \tag{7}$$

The $\pi/4$-property (2) generalizes to

$$\operatorname{Re}\left\{\frac{e^{-i\alpha}(w-B)}{B(B-1)}\right\} > 0, \quad w \in \Gamma, \; w \neq 0. \tag{8}$$

Because $0 \leq \alpha \leq \pi$, it is geometrically clear that we may assume $\operatorname{Im}\{B\} \geq 0$.

For what values of α and ζ is it possible for the omitted arc Γ to be a linear ray? Substitution of the curve $w = e^{i\gamma} t$ into (7) gives

$$\frac{e^{i(\alpha+\gamma)} B(B-1)}{(e^{i\gamma}t - 1)(e^{i\gamma}t - B)} > 0, \quad t \geq 0.$$

For $t = 0$ this implies

$$e^{i(\alpha+\gamma)}(B-1) > 0, \tag{9}$$

so that

$$(e^{i\gamma} t - 1)(B^{-1} e^{i\gamma} t - 1) > 0, \; t \geq 0. \tag{10}$$

Now let $t \to \infty$ in (10) to obtain

$$e^{2i\gamma} = B / |B|. \tag{11}$$

Set $t = |B|$ in (10) and use (11) to conclude that

$$(e^{i\gamma} |B| - 1)(e^{-i\gamma} - 1) > 0. \tag{12}$$

There are now two cases.

Case I: $e^{i\gamma} = -1$. Then $B > 0$, by (11). It follows from (9) that either $e^{i\alpha} = 1$ and $0 < B < 1$, or $e^{i\alpha} = -1$ and $B > 1$. The choice $e^{i\alpha} = 1$ has been treated in the previous sections and, at least for $0 < \zeta \leq 3 - \sqrt{8}$, has $k_0(-z)$ as its extremal function. The choice $e^{i\alpha} = -1$, as mentioned earlier, is equivalent to the maximum modulus problem and is solved for all $\zeta, 0 < \zeta < 1$, by $k_0(z)$.

Case II: $e^{i\gamma} \neq -1$. Since $e^{i\gamma} \neq 1$, this means that $\operatorname{Im}\{e^{i\gamma}\} \neq 0$, so that (12) implies $|B| = 1$. Thus $B = e^{2i\gamma}$, and (9) gives $ie^{i\alpha} B \sin\gamma > 0$. Therefore, either $\sin\gamma > 0$ and $B = -ie^{-i\alpha}$, or $\sin\gamma < 0$ and $B = ie^{-i\alpha}$. In the other hand, (6 a) shows $\operatorname{Re}\{e^{i\alpha} B\} < 0$ if $\pi/2 \leq \alpha \leq \pi$. Thus $0 \leq \alpha < \pi/2$, and the requirement that $\operatorname{Im}\{B\} > 0$ eliminates the possibility that $B = -ie^{-i\alpha}$. We conclude that $B = ie^{-i\alpha}$ and $e^{i\gamma} = -e^{i(\pi/4 - \alpha/2)}$.

We will show presently that some choice of ζ actually produces $B = ie^{-i\alpha}$ as the value of an extremal function for the problem (6), where $0 \leq \alpha < \pi/2$. It will then follow from what we have just observed that this extremal function f maps $\mathbb{D}$ onto the complement of a linear ray in the direction $-e^{i(\pi/4 - \alpha/2)}$. Consequently, f must have the form

$$f(z) = \left[\frac{1 + e^{i(\pi/4 - \alpha/2 + \beta)} z}{1 - e^{i\beta} z}\right]^2, \tag{13}$$

where the rotation factor $e^{i\beta}$ is determined by the condition $f(\zeta) = ie^{-i\alpha}$. A straightforward calculation leads to the values $e^{i\beta} = e^{i(3\pi/8 + \alpha/4)}$ and $\zeta = \sin(\pi/8 - \alpha/4)$. These results are summarized by the following theorem.

Theorem 1. *For each $f \in S_0$ and for $0 \leq \alpha < \pi/2$, the inequality* $\operatorname{Re}\{e^{i\alpha} f(z)\} \geq 0$ *holds in the disc $|z| \leq \sin(\pi/8 - \alpha/4)$, and this radius is sharp for each α. If $z = \zeta = \sin(\pi/8 - \alpha/4)$ and $0 < \alpha < \pi/2$, then* $\operatorname{Re}\{e^{i\alpha} f(\zeta)\} = 0$ *if and only if f is the function* (13), *with $\beta = 3\pi/8 + \alpha/4$ and $f(\zeta) = ie^{-i\alpha}$. If $z = \zeta = \sin\pi/8$, then* $\operatorname{Re}\{f(\zeta)\} = 0$ *if and only if f is either function* (13), *with $\alpha = 0$ and $\beta = 3\pi/8$ and $f(\zeta) = i$, or its conjugate function $\overline{f(\bar{z})}$.*

For $\alpha = 0$ this theorem is equivalent to Theorem 7 in [1]. There is a curious geometric corollary.

Corollary 1. *For $0 \leq \zeta < \sin(\pi/8) = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, the region of values*

$$W_\zeta = \{f(\zeta) : f \in S_0\}$$

lies in the right half-plane Re $\{w\} > 0$. *Each supporting line through the origin meets* W_ζ *at exactly one point, and this point has unit modulus.*

Among the solutions to a linear problem such as (6) there must be an extreme point of S_0. Therefore, on the basis of Theorem 1, we can identify some extreme points in addition to $k_0(e^{i\phi}z), 0 \leqslant \phi < 2\pi$.

Corollary 2. *For* $0 \leqslant \alpha < \pi/2$, *let* $f(z)$ *be given by* (13) *with* $\beta = 3\pi/8 + \alpha/4$. *Then the functions defined by* $f(e^{i\phi}z)$ *and* $\overline{f(e^{i\phi}\overline{z})}$ *for* $0 \leqslant \phi < 2\pi$ *are extreme points of* S_0.

The proof of Theorem 1 is contingent upon a demonstration that $B = ie^{-i\alpha}$ actually occurs as the value of an extremal function for the problem (6) for some choice of ζ. As in the case where $\alpha = 0$, the condition Re $\{e^{i\alpha}B\} \geqslant 0$ assures that the point 1 lies outside the half-plane (8) containing the omitted arc Γ, so Γ cannot cross the line segment joining 1 to B. It follows that B must satisfy $e^{i\alpha/2}J(B) > 0$, where J is expressed by the integral (3). Thus the desired result is a consequence of the following theorem, a generalization of Theorem 9 in [1].

Theorem 2. *For* $0 \leqslant \alpha < \pi/2$, *the integral* $J(B)$ *given in* (3) *has the properties*

$$e^{i\alpha/2}J(ie^{-i\alpha}) > 0,$$

$$\operatorname{Im}\left\{e^{i\alpha/2}J(ibe^{-i\alpha})\right\} > 0 \quad \text{for } 0 < b < 1,$$

and

$$\operatorname{Im}\left\{e^{i\alpha/2}J(ibe^{-i\alpha})\right\} < 0 \quad \text{for } 1 < b < \infty.$$

Corollary. *The condition* $e^{i\alpha/2}J(B) > 0$ *is satisfied at* $B = ie^{-i\alpha}$ *and at no other point on the ray* $B = ibe^{-i\alpha}, 0 < b < \infty$. *Thus* $B = ie^{-i\alpha} = f(\zeta)$ *for some* ζ *in the interval* $0 < \zeta < 1$, *where* f *is an extremal function for the corresponding problem* (6).

Proof of Theorem 2. The proof is similar to that of Theorem 9 in [1], with a few simplifications. The substitution $t = 1 - s$ for $\frac{1}{2} \leqslant t \leqslant 1$ reduces the integral to

$$J(B) = \int_0^{1/2}\left\{\left[\frac{B(1-B)}{1-t(1-B)}\right]^{1/2} + \left[\frac{B(1-B)}{B+t(1-B)}\right]^{1/2}\right\}\frac{dt}{\sqrt{t(1-t)}}.$$

A calculation gives

$$e^{i\alpha/2}J(ibe^{-i\alpha}) = \sqrt{b}\int_0^{1/2}\left\{[H_1]^{1/2} + [H_2]^{1/2}\right\}\frac{dt}{\sqrt{t(1-t)}},$$

where

$$H_1 = H_1(t, b, \alpha) = \frac{b\cos\alpha + i\,[(1-t) - tb^2 + (2t-1)\,b\sin\alpha]}{(1-t)^2 + t^2b^2 + 2t(1-t)\,b\sin\alpha}$$

and

$$H_2 = H_2(t, b, \alpha) = \frac{b\cos\alpha + i\,[(1-t)\,b^2 - t + (2t-1)\,b\,\sin\alpha]}{(1-t)^2 b^2 + t^2 + 2t(1-t)\,b\,\sin\alpha}.$$

It is now apparent that $H_1(t, 1, \alpha) = H_2(t, 1, \alpha)$ and that $e^{i\alpha/2} J(ie^{-i\alpha}) > 0$.

Now let $[H_j]^{1/2} = (x_j + iy_j)^{1/2} = \xi_j + i\eta_j$. For the remainder of the proof it is sufficient to show that $\eta_1 > \eta_2$ in the interval $0 < t < 1/2$ if $0 < b < 1$, while $\eta_1 < \eta_2$ if $1 < b < \infty$. Since $H_1(t, 1/b, \alpha) = H_2(t, b, \alpha)$, it is enough to consider the case $0 < b < 1$. Then $0 < x_1 < x_2$, an inequality equivalent to $(1-2t)(1-b^2) > 0$. Another calculation shows that the inequality $y_2 < y_1$ is equivalent to

$$t(1-t)(1-b^4) + [(1-2t) + 2t^2]\,(1-b^2)\,b\sin\alpha > 0,$$

which is obviously true. Observe next that $y_1 > 0$ because

$$(1 - b\sin\alpha) + (2b\sin\alpha - b^2 - 1)\,t$$

is a linear function of t which is positive both for $t = 0$ and for $t = 1/2$. If $y_2 \leq 0$, then it is obvious that $\eta_1 > \eta_2$. If $y_2 > 0$, then we have $0 < x_1 < x_2$ and $0 < y_2 < y_1$, so that a simple graphical argument (cf. [1], p. 213) shows that $\eta_1 > \eta_2$. This completes the proof.

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STRESZCZENIE

Autorzy rozważają problem wyznaczenia min $\{\mathrm{Re}\, f(z)\}$ w klasie funkcji holomorficznych i jednolistnych w kole $|z| < 1$ i unormowanych przez warunki: $f(z) \neq 0$, $f'(0) = 1$.

РЕЗЮМЕ

Авторы изучают проблему min $\{\mathrm{Re}\, f(z)\}$ в классе голоморфных и однолистных функций в единичном круге $|z| < 1$ нормированных через условия: $f(z) \neq 0$, $f(0) = 1$.