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On Extreme Points and Support Points of the Class S

O punktach ekstremalnych i punktach podpierających dla klasy S

Об экстремальных и опорных точках класса S

Let S be the usual class of normalized univalent functions on the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in S$ is called a support point of S if there exists a continuous linear functional J on the space $H(\Delta)$ of holomorphic functions on Δ such that

$$\operatorname{Re} J(h) \leq \operatorname{Re} J(f) \quad \text{for all } h \in S \tag{1}$$

and

$$\operatorname{Re} J(h) < \operatorname{Re} J(f) \quad \text{for some } h \in S. \tag{2}$$

If $\mathbf{S}(S)$ denotes the set of all support points of S, and $E(\overline{\text{co}}\, S)$ the set of extreme points of the closed convex hull of S, it is well known that $\overline{\text{co}}\, S$ is compact and that $E(\overline{\text{co}}\, S) \subset S$ [3, p. 440], but it is not known either whether $E(\overline{\text{co}}\, S) \subset \mathbf{S}(S)$ or $\mathbf{S}(S) \subset E(\overline{\text{co}}\, S)$. In this paper we shall prove the second inclusion except for certain special support points which we call terminal support points of S. These are defined in the next paragraph.

Albert Pfluger [8] and later L. Brickman and D. R. Wilken [2] showed that if $f \in \mathbf{S}(S)$, then $\mathbb{C} \setminus f(\Delta)$ is a single analytic arc extending to ∞. Pfluger also proved that if a (half closed) subarc is removed from $\mathbb{C} \setminus f(\Delta)$, beginning at the finite tip, then the resulting region-after being contracted by a suitable numerical factor-again corresponds to a support point of S. (We include a simple proof below.)

Definition. A support point of S obtained by the procedure just described will be said to be obtained by arc truncation. More explicitly: the support point g is obtained from the support point f by arc truncation if for some number $r > 1$, $f \precneqq rg$ (f is subordinate to rg). A support point of S that can be obtained by arc truncation is called a nonterminal support point; one that cannot be so obtained is called a terminal support point.

Remarks. (i) Stated in reverse a terminal support point is one whose omitted arc cannot be lengthened to produce-after normalization by a factor greater than 1-another support point. It may be mentioned, however, that the omitted arc of any support point of S admits an analytic extension [2, Lemma 5].

(ii) A Koebe function is nonterminal because it can be obtained by truncation of its own omitted arc.

(iii) Examples of terminal support points can be found among the support points recently discovered by K. Pearce [6]. The omitted arcs of his terminal support points are half-lines making an angle of $\pi/4$ with the radius vector to the tip.

(iv) An interesting question is whether every nonterminal support point of S is obtainable from some terminal support point by arc truncation. In other words do the terminal support points generate all others? (Actually the Koebe functions would have to be obtained as limits as the length of arc removed becomes infinite. That such a limit always produces a Koebe function was by shown Pfluger [8], who made use of the fact that the omitted arc of any support point of S has an asymptotic line at ∞.) In any case terminal support points appear to be rather special and relatively rare in $\mathbf{S}(S)$, for each such support point is 'at the base of' an uncountable family of nonterminal support points.

Very recently W. E. Kirwan and G. Schober [5], one of the present authors (unpublished), and perhaps others have found easy proofs of Pfluger's result that arc truncation preserves support points of S. (It should be mentioned that [5] treats nonlinear functionals as well as linear ones, and classes other than S as well as S.) Since some of these recent proofs have not made (2), the nonconstancy requirement for support points, sufficiently clear, and since Pfluger's result admits an easy generalization (Proposition 1) which may prove useful, we present a proof here (Proposition 2).

Proposition 1. *Let T be a continuous linear operator on $H(\Delta)$ such that $T(S) \subset S$ but $T(S) \not\subset \mathbf{S}(S)$. Then*

$$g \in S,\ T(g) \in \mathbf{S}(S) \Longrightarrow g \in \mathbf{S}(S).$$

Proof. Let $T(g) = f$ and let J be a continuous linear functional related to f as in (1) and (2). Define the continuous linear functional K by $K = J \cdot T$. Then for any $h \in S$ there follows $T(h) \in S$ and hence, by (1),

$$\operatorname{Re} K(h) = \operatorname{Re} J(T(h)) \leqslant \operatorname{Re} J(f) = \operatorname{Re} J(T(g)) = \operatorname{Re} K(g).$$

Thus g and K satisfy (1). To prove (2) for g and K we choose $h \in S$ such that $T(h) \notin \mathbf{S}(S)$. Then $\operatorname{Re} J(T(h)) < \operatorname{Re} J(f)$, that is $\operatorname{Re} K(h) < \operatorname{Re} K(g)$ as required.

Proposition 2. (Pfluger). *Let $f \in \mathbf{S}(S)$ and let $g \in S$ be obtained from f by arc truncation. Then $g \in \mathbf{S}(S)$.*

Proof. The hypothesis means that $f \lneqq rg$ for some $r > 1$. We define ϕ by the equation $f = rg \cdot \phi$ and then the operator T on $H(\Delta)$ by $T(h) = rh \cdot \phi$. It is easy to verify that $T(S) \subset S$. Also, if $h(z) = z$ (the identity function), then $T(h) = r\phi$. This function is bounded and therefore not a support point of S. Thus $T(S) \not\subset \mathbf{S}(S)$. Finally, since $Tg = f \in \mathbf{S}(S)$, the desired conclusion follows at once from Proposition 1.

We now state our theorem. We do not know whether 'nonterminal' can be eliminated.

Theorem. *Every nonterminal support point of S is an extreme point of* $\overline{\text{co}}\, S$.

Proof. Let g be a support point of S obtained from the support point f by arc truncation. We express the relationship between f and g explicitly as follows. Let f be imbedded in a Loewner chain $F(z, t)$:

$$F(z, t) = e^t z + \dots, \quad F(z, 0) = f(z) \quad (z \in \Delta, 0 \leqslant t < \infty). \tag{3}$$

(See [9. pp. 156–164] for the required information concerning subordination chains.) Then for some $\tau > 0$ we have

$$g(z) = e^{-\tau} F(z, \tau) \quad (z \in \Delta) \tag{4}$$

We must show $g \in E(\overline{\text{co}}\, S)$.

Associated with the chain $F(z, t)$ there is a 'subordinating function' $\omega(z, s, t)$ satisfying the conditions

$$\omega(z, s, t) = e^{s-t} z + \dots, \quad |\omega(z, s, t)| \leqslant |z| \quad (z \in \Delta, 0 \leqslant s \leqslant t < \infty), \tag{5}$$

$$F(z, s) = F(\omega(z, s, t), t) \quad (z \in \Delta, 0 \leqslant s \leqslant t \leqslant \infty), \tag{6}$$

$$\omega(z, s, \tau) = \omega(\omega(z, s, t), t, \tau) \quad (z \in \Delta, 0 \leqslant s \leqslant t \leqslant \tau). \tag{7}$$

We now define

$$\phi(z) = \omega(z, 0, \tau) \quad (z \in \Delta) \tag{8}$$

and obtain, as a special of (6),

$$F(z, 0) = F(\phi(z), \tau) \quad (z \in \Delta).$$

By (3) and (4) this becomes

$$f(z) = e^{\tau} g(\phi(z)) \quad (z \in \Delta). \tag{9}$$

Next we employ Choquet's theorem [7, pp. 19–20] to obtain a probability measure μ on $E(\overline{\text{co}}\, S)$ such that

$$\text{Re}\, L(g) = \int \text{Re}\, L(h)\, d\mu(h) \tag{10}$$

for every continuous linear functional L on $H(\Delta)$. We now let J be a functional associated with f as in (1) and (2), and choose

$$L(h) = J(e^{\tau} h \circ \phi) \quad (h \in H(\Delta)). \tag{11}$$

By (9), (10) then becomes

$$\operatorname{Re} J(f) = \int \operatorname{Re} J(e^{\tau} h \cdot \phi)\, d\mu(h). \tag{12}$$

The function $e^{\tau} h \cdot \phi$ in the integrand is univalent because ϕ is univalent and $E(\overline{\text{co}}\, S) \subset S$. Moreover by (5) and (8), $e^{\tau} h \cdot \phi$ has the required normalization for S. Hence, by (1),

$$\operatorname{Re} J(e^{\tau} h \cdot \phi) \leqslant \operatorname{Re} J(f) \quad h \in E(\overline{\text{co}}\, S).$$

Thus, writing (12) as

$$\int [\operatorname{Re} J(f) - \operatorname{Re} J(e^{\tau} h \cdot \phi)]\ d\mu(h) = 0,$$

and noting that the integrand in continuous, we can conclude that $\operatorname{Re} J(f) = \operatorname{Re} J(e^{\tau} h \cdot \phi)$ for every $h \in E(\overline{\text{co}}\, S)$ in the support of the measure μ. (We say h is the support of μ if $\mu(V) > 0$ for every neighborhood V of h.) Thus we can choose a function h such that

$$h \in E(\overline{\text{co}}\, S),\ e^{\tau} h \cdot \phi \in \mathrm{S}(S). \tag{13}$$

We shall complete the proof by showing that the second condition in (13) implies that $h = g$. Then $g \in E(\overline{\text{co}}\, S)$ as required.

For h satisfying (13) we define the finite subordination chain $G(z, t)$ by

$$G(z, t) = e^{\tau} h(\omega(z, t, \tau)) \quad (z \in \Delta, 0 \leqslant t \leqslant \tau), \tag{14}$$

and note

$$G(z, t) = e^{t} z + \ldots\ (z \in \Delta, 0 \leqslant t \leqslant \tau) \tag{15}$$

and

$$G(z, 0) = e^{\tau} h(\phi(z)) \in \mathrm{S}(S). \tag{16}$$

Applying h to equation (7) and multiplying the result by e^{τ} we obtain, by (14),

$$G(z, s) = G(\omega(z, s, t), t) \quad (z \in \Delta, 0 \leqslant s \leqslant t \leqslant \tau). \tag{17}$$

Thus $G(z, t)$ is a normalized subordination chain with the some subordinating function, namely $\omega(z, s, t)$, as the Loewner chain $F(z, t)$. (Compare (6) and (17).) Finally we imbed $G(z, 0)$ in a Loewner chain $H(z, t)$:

$$H(z, t) = e^{t} z + \ldots,\ H(z, 0) = G(z, 0) \quad (z \in \Delta, 0 \leqslant t < \infty). \tag{18}$$

For $0 \leq t \leq \tau$ and $z \in \Delta$ we must have $H(z, t) = G(z, t)$. Indeed for any t satisfying $0 \leq t \leq \tau$, the mapping $G(z, 0)$ is subordinate to both $G(z, t)$ and $H(z, t)$. But $G(z, 0)$, by (16), is a slit mapping. It follows that one of the mappings $G(z, t), H(z, t)$ is subordinate to the other. But both are normalized. (Derivative at 0 equals e^t.) Thus these mappings must be the same. We can now replace (17) by

$$H(z, s) = H(\omega(z, s, t), t) \quad (z \in \Delta, 0 \leq s \leq t \leq \tau). \tag{19}$$

In the appendix we shall show that (19) leads by analytic continuation to

$$H(z, s) = H(\omega(z, s, t), t) \quad (z \in \Delta, 0 \leq s \leq t < \infty). \tag{20}$$

Intuitively, the fact that the omitted arc of $G(z, 0)$ is analytic implies that the Loewner chain $H(z, t)$ of (18) is analytic in both variables. Similarly the omitted arc of $F(z, 0)$ is analytic, so the function $\omega(z, s, t)$ is analytic in all three variables. Hence (19) implies (20). Thus, by (6) and (20), the Loewner chains $F(z, t)$ and $H(z, t)$ have the same subordinating function $\omega(z, s, t)$. Therefore these chains must be identical. Indeed

$$F(z, t) = \lim_{u \to \infty} e^u \omega(z, t, u) = H(z, t) \quad (z \in \Delta, 0 \leq t < \infty).$$

In particular $F(z, \tau) = H(z, \tau) = G(z, \tau) = e^\tau h(\omega(z, \tau, \tau)) = e^\tau h(z)$. (See (5).) Therefore $h(z) = e^{-\tau} F(z, \tau) = g(z)$ as claimed.

APPENDIX

The following theorem provides the final step in the proof of the main theorem in the text. It will be used to show how equation (20) follows from equation (19).

Theorem A: *Let $f(z) \in S$ be such that $\mathbb{C} \setminus f(\Delta) = \gamma$ is a single arc, analytic everywhere including at the base point and at ∞. Then there exists a normalized subordination chain $F(z, t)$ with $F(z, 0) = f(z)$ and such that $F(z, t)$ is analytic in t.*

Proof: By the analyticity of γ there exist open sets U and V and a function w such that

i) $U \supset \{\zeta = x + iy: x \geq 0, y = 0\}$ and $\mathbb{C} \setminus U$ is compact,

ii) $V \supset \gamma$ and $\mathbb{C} \setminus V$ is compact,

iii) $w: U \to V$ is one-to-one, onto, and analytic with $w(0) =$ base point of γ.

Also without loss of generality we can assume there exist $\epsilon > 0, R > 0$ such that $U = \{\zeta = x + iy: x > -\epsilon, |y| < \epsilon \cup \zeta: |\zeta| > R\}$. Consider the 'variation' given by

$$z^* = z^*(z, t) = w\left[w^{-1}(z) + t\right], \; t > 0, \; z \in V.$$

Then z^* is an analytic function of both z and t and, expressed as a power series in t, looks like

$$z^* = z + tV_1(z) + t^2 V_2(z) + \ldots.$$

For a fixed τ consider the arc $\gamma_\tau = w\,[w^{-1}(\gamma) + \tau]$ and the associated domain $\mathbb{C} \setminus \gamma_\tau$. In [4] it is shown that, using a variation such as the one given above, if one considers the analytic completion of the Green's functions associated with the varied domains then these functions vary analytically with t and consequently so do the associated mapping functions. That is, if we let $\tilde{F}(z, t)$ denote the associated mapping function for a given t, then $\tilde{F}(z, t)$ is a subordination chain such that $\tilde{F}(z, 0) = f(z)$ and $\tilde{F}(z, t)$ is analytic in t. If we write $\tilde{F}(z, t) = \eta(t)\, z + \ldots$ Then the analyticity and the strict subordination imply that $\eta(t)$ has an analytic inverse.

Define $F(z, t) = \tilde{F}(z, \eta^{-1}(e^t))$. Then $F(z, t) = e^t z + \ldots$ and $F(z, t)$ is the desired normalized subordination chain.

Finally to obtain equation (20) it suffices to show for each fixed z, $\omega(z, s, t)$ is analytic in both s and t. Recall (6) which states

$$F(z, s) = F(\omega(z, s, t), t)\,.$$

For any $f(z)$ analytic and univalent in Δ and any R, $0 < R < 1$, if $w = f(z)$ and $|z| < R$, then ([1], p. 153)

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{\zeta f'(\zeta)}{f(\zeta) - w}\, d\zeta$$

If we set $w = F(z, s)$ then (6) implies

$$\omega(z, s, t) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{\zeta F'(\zeta, t)\, d\zeta}{F(\zeta, t) - F(z, s)}\,, \quad |z| < R$$

and the result follows.

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STRESZCZENIE

Niech S oznacza klasę unormowanych funkcji holomorficznych i jednolistnych w kole jednostkowym, S (S) – zbiór punktów podpierających klasy S zaś E ($\overline{co}\, S$) zbiór punktów ekstremalnych domkniętej otoczki wypukłej klasy S.

Autorzy dowodzą, że wyjąwszy punkty podpierające pewnego specjalnego typu, ma miejsce inkluzja S (S) $\subset$ E ($\overline{co}\, S$).

РЕЗЮМЕ

Пусть S класс нормированных, голоморфных и однолистных функций в единичном круге, S (S) – множество опорных точек класса S, E ($\overline{co}\, S$) – множество экстремальных точек замкнутой, выпуклой оболочки класса S.

Авторы доказывают, что исключая опорные точки некоторого специального типа, имеет место включения S (S) $\subset$ E ($\overline{co}\, S$).