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A Distortion Theorem for Close-to-Convex Functions

Pewne twierdzenie o zniekształceniu dla funkcji prawie wypukłych

Abstract. In this paper the exact value of $\max\left|\frac{z}{f(z)}-1\right|$ in the class C for fixed z in the unit disk is obtained.

Let S denote the class of functions $f(z)=z+a_2z^2+\ldots$ analytic and univalent in the unit disk $\mathbf{D}=\{z:|z|<1\}$. We denote by S^* and C the subclasses of S of starlike and close-to-convex functions respectively. We shall prove the following theorem.

Theorem 1. *Let $|z|=r<1$ and $f\in C$. Then*

(1) $$\left|\frac{z}{f(z)}-1\right|\leq 2r+r^2$$

Causey, Krzyż and Merkes proved (1) for $f\in S^*$ ([2]). They also showed that if $f\in S$ then $|z/f(z)-1|\leq |a_2|r+3r^2$ ([2]).

Proof. We shall use the following result of Biernacki [1]:

Let $\Omega_r(C)=\left\{\frac{z}{f(z)}:|z|\leq r<1\,,\,f\in C\right\}$

and $\Gamma_r(C)=\left\{\frac{zf'(z)}{f(z)}:|z|\leq r<1\,,\,\in C\right\}$

then

(2) $$\Omega_r(C)=\left\{u=\frac{(1+s)^2}{1+\frac{1}{2}(s+t)}:|s|\leq r\,,\,|t|\leq r\right\}$$

Biernacki also showed that

(3) $$\Gamma_r(C)=\frac{1}{1-r^2}\,\Omega_r(C)\,.$$

By (2) we have for $|z|\leq r<1$ and $f\in C$

$$\left|\frac{z}{f(z)}-1\right|\leq\sup\left\{\left|\frac{(1+s)^2}{1+\frac{1}{2}(s+t)}-1\right|:|s|\leq r\,,\,|t|\leq r\right\}=$$

$$=\sup\left\{\left|\frac{2s^2+3s-t}{2+s+t}\right|:|s|\leq r\,,\,|t|\leq r\right\}.$$

We can take $|s| = |t| = r$ because of the maximum principle. Let s be fixed. The homography $h : t \mapsto \dfrac{2s^2+3s-t}{2+s+t}$ maps the disk $\{t : |t| \le r\}$ onto the disk of center $w = \dfrac{s(2s+3)(\bar{s}+2)+r^2}{|s+2|^2-r^2}$ and radius $R = 2r\,\dfrac{|s+1|^2}{|s+2|^2-r^2}$. Hence

$$\left|\frac{z}{f(z)}-1\right| \le \sup|w+R| \le \sup|w| + \sup R\ .$$

We have

$$|w| = \frac{|s(2s+3)(\bar{s}+2)+r^2|}{|s+2|^2-r^2} = \frac{|s(2s+3)(\bar{s}+2)+s\bar{s}|}{|s+2|^2-r^2} = \frac{r}{2}\,\frac{|r^2+3+4\,\mathrm{Re}\,s|}{1+\mathrm{Re}\,s} =$$

$$= \frac{r}{2}\left(4-\frac{1-r^2}{1+\mathrm{Re}\,s}\right) \le \frac{r}{2}\left(4-\frac{1-r^2}{1+r}\right) = \frac{r}{2}(3+r)\,,$$

$$R = 2r\,\frac{|s+1|^2}{|s+2|^2-r^2} = \frac{r}{2}\,\frac{r^2+1+2\,\mathrm{Re}\,s}{1+\mathrm{Re}\,s} = \frac{r}{2}\left(2-\frac{1-r^2}{1+\mathrm{Re}\,s}\right) \le \frac{r}{2}(1+r)\ .$$

Hence

$$\left|\frac{z}{f(z)}-1\right| \le r^2+2r$$

and the proof is completed.

Let M denote the class of functions $f(z) = z + a_2z^2 + \cdots$ analytic and univalent in the unit disk $\mathbf{D} = \{z : |z| < 1\}$, defined by

$$f \in M \iff (f * g) \neq 0$$

for every $g \in S^*$. The class M satisfies following inclusions

$$C \subset M \subset S$$

Rønning showed [4] that $\Gamma_r(C) = \Gamma_r(M)$. By (3) we have also $\Omega_r(C) = \Omega_r(M)$ and Theorem 1 remains true if we replace C by M.

Corollary .Let $|z| = r < 1$ and $f \in M$. Then

$$\left|\frac{z}{f(z)}-1\right| \le 2r+r^2\,. \tag{4}$$

Let $F(z) = \int_0^z (f'(t)^\alpha \left(\frac{g(t)}{t}\right)^\beta dt$ where $f \in S$ and $g \in S$. Miazga and Wesołowski [3] gave a sufficient condition on α and β for univalence of $F(z)$ in $\mathbf{D}$, but in the proof they used an incorrect statement that $|z/f(z)-1| \le 2r+r^2$ for $f \in S$ and $|z| \le r < 1$. However, under modified assumptions we will get a similar but weaker result

Theorem 2. *Let $F(z) = \int_0^z (f'(t)^\alpha \left(\frac{g(t)}{t}\right)^\beta dt$ where $f \in S$ and $g \in M$. Then $F(z)$ is univalent in $\mathbf{D}$ for each α and β such that $4|\alpha| + 3|\beta| \le 1$*

Proof is based on Ahlfors univalence criterion and is the same as in [3].

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STRESZCZENIE

W pracy podano dokladne oszacowanie funkcjonalu $\left|\frac{z}{f(z)} - 1\right|$ dla $|z| = r < 1$ i $f \in C$.

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