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Approximation of the Hersch–Pfluger Distortion Function. Applications

Aproksymacja funkcji zniekształcenia Herscha–Pflugera. Zastosowania

Abstract This paper aims at giving a number of applications of an approximation method of the distortion function Φ_K. As a result some new bounds for the functions μ^{-1}, λ and μ are established. Moreover, the error in the approximation of functions mentioned above is given which is helpful for numerical calculations.

0. Introduction. In the theory of plane quasiconformal mappings the function Φ_K defined as follows

$$\Phi_K(r) = \mu^{-1}(\mu(r)/K)\ , \quad 0 < r < 1\ ,\ K > 0\ ,\ \Phi_K(0) = 0\ ,\ \Phi_K(1) = 1\ , \tag{0.1}$$

plays an important role. Here μ stands for the module of the Grötzsch extremal domain $\mathbf{B}^2 \setminus [0, r]$ and it can be expressed in the form

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}\ , \quad 0 < r < 1\ , \tag{0.2}$$

where

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}\ , \quad 0 < r < 1\ ,$$

is the elliptic integral of the first kind. The function Φ_K is called the Hersch–Pfluger distortion function, cf. [4], and it was studied by many mathematicians. Recently Anderson, Vamanamurthy, Vuorinen, cf. [1], [9], [10], and Zając, cf. [11], [12] obtained many interesting results concerning the properties of the function Φ_K and its estimates.

In the paper [8] the functions $\varphi_{K,t}$, $\widetilde{\varphi}_{K,t}$, $\psi_{K,t}$, $\widetilde{\psi}_{K,t}$, depending on a real parameter $t \geq 1$ were introduced, see (1.1) and (1.3). It turns out that these functions are monotonically convergent to Φ_K as $t \to \infty$, cf. [8, Theorem 1.3, Corollary 1.4], and the convergence to the function Φ_K is very fast, cf. [8, Theorem 1.5, Corollary 1.6]. In the section 1 of this paper we complete those considerations. We also examine a pair of sequences $\varphi_{K,t}$, $\widetilde{\varphi}_{K,t}$, $t = 2^n$, $n = 0, 1, 2, \ldots$ and we establish Theorem 1.2 and Corollary 1.3 which give their deviation from the function Φ_K. Since the function μ^{-1} and the distortion function λ, cf. [6], [5], introduced by Lehto, Virtanen and Väisälä in [7] can be expressed by means of the function Φ_K, see (2.1) and (2.3), we can apply results from the first section to approximate them, cf. Corollary 2.1. This way we establish in the second section new upper and lower bounds for the functions μ^{-1}, cf. Theorem 2.2, and λ, cf. Corollaries 2.3, 2.4 which improve some recent results obtained by Anderson, Vamanamurthy, Vuorinen [1], [9], [10]. We also apply results from the first section to estimate the module function μ but in a slightly different way. Theorem 2.5 improves some results from [1], [2], [3], [9], [10]. The last section 3 is devoted to numerical applications of Theorems 1.1 and 1.2. We study the error of the approximating sequences $\psi_{K,2^n}$, $\widetilde{\psi}_{K,2^n}$, $n = 0, 1, 2, \ldots$ for the functions Φ_K, μ^{-1} and λ. It turns out, in view of Theorem 3.1 and Corollaries 3.2, 3.3 that the approximation methods mentioned above can be used for the calculation of the values of Φ_K, μ^{-1} and λ by a computer. It is worth mentioning that the techniques developed here and in [8] are alternative to those used by Anderson, Vamanamurthy and Vuorinen in papers refered to.

1. The main approximation results. In the paper [8] the following functions $\varphi_{K,t}$, $\psi_{K,t}$, $\widetilde{\varphi}_{K,t}$ and $\widetilde{\psi}_{K,t}$, $t \geq 1$, were introduced. We remind below their definitions for the convenience of the reader.

$$\varphi_{K,t} = \Phi_t \circ \varphi_{K,1} \circ \Phi_{1/t} \quad \text{and} \quad \psi_{K,t} = \Phi_t \circ \psi_{K,1} \circ \Phi_{1/t} \tag{1.1}$$

for any $K > 0$ and $t \geq 1$ where

$$\varphi_{K,1}(x) = x^{1/K} \quad \text{and} \quad \psi_{K,1}(x) = \min\{4^{1-1/K} x^{1/K}, 1\}, \quad 0 \leq x \leq 1\,. \tag{1.2}$$

and

$$\widetilde{\varphi}_{K,t} = h \circ \varphi_{1/K,t} \circ h\,, \ \widetilde{\psi}_{K,t} = h \circ \psi_{1/K,t} \circ h\,, \quad K > 0,\ t \geq 1 \tag{1.3}$$

where $h(x) = (1-x)(1+x)^{-1}$, $0 \leq x \leq 1$.

Some important facts about these functions were established in [8, Theorems 1.3, 1.5 and Corollaries 1.4, 1.6], but for the convenience of the reader we collect them into the following

Theorem 1.1. *For every $K \geq 1$ ($0 < K < 1$) the functions $\varphi_{K,t}(x)$, $\widetilde{\varphi}_{K,t}(x)$ of a real parameter $t \geq 1$ are increasing (decreasing), resp., whereas $\psi_{K,t}(x)$, $\widetilde{\psi}_{K,t}(x)$ are decreasing (increasing), resp., where x is any fixed number between 0 and 1. Moreover, for any $0 < x < 1$ and $K > 0$*

$$\lim_{t\to\infty} \varphi_{K,t}(x) = \lim_{t\to\infty} \widetilde{\varphi}_{K,t}(x) = \lim_{t\to\infty} \psi_{K,t}(x) = \lim_{t\to\infty} \widetilde{\psi}_{K,t}(x) = \Phi_K(x)$$

and the following estimates hold

$$(1 - x^{2^{n+1}/K})\psi_{K,2^n}(x) \leq \Phi_K(x) \leq \psi_{K,2^n}(x)\,, \quad n = 2, 3, 4, \ldots \tag{1.4}$$

(1.5) $0 \le \widetilde{\psi}_{K,2^n}(x) - \Phi_K(x) \le 2((1-h(x)^{2^{n+1}})^{-K2^{-n}} - 1)h^K(x)\,, \quad n = 1,2,3,\dots$

as $K \ge 1$ *and*

(1.6) $$\psi_{K,2^n}(x) \le \Phi_K(x) \le (1 - x^{2^{n+1}})^{-1/K2^n}\psi_{K,2^n}(x)\,, \quad n = 1,2,3,\dots .$$

(1.7)
$$0 \le \Phi_K(x) - \widetilde{\psi}_{K,2^n}(x) \le 2((1-h(x)^{K2^{n+1}})^{-1}-1)\min\{4^{1-K}h^K(x), 1\},\ n=2,3,4,\dots$$

as $0 < K \le 1$.

Now we shall prove a result complementary to [8, Theorem 1.5]

Theorem 1.2. *For any* $0 \le x \le 1$ *and* $n = 0,1,2,\dots$

(1.8) $$\varphi_{K,2^n}(x) \le \psi_{K,2^n}(x) \le 4^{(1-1/K)2^{-n}}\varphi_{K,2^n}(x) \quad \text{as} \quad K \ge 1$$

and

(1.9) $$4^{(1-1/K)2^{-n}}\varphi_{K,2^n}(x) \le \psi_{K,2^n}(x) \le \varphi_{K,2^n}(x) \quad \text{as} \quad 0 < K < 1$$

Proof. Let $K \ge 1$ be fixed. Obviously for any $0 < x \le 1$

(1.10) $$\frac{\psi_{K,1}(x)}{\varphi_{K,1}(x)} \le 4^{1-1/K}\,.$$

If $n = 0,1,2,\dots$ is arbitrary then by Theorem 1.1 and the equality

(1.11) $$\Phi_2(r) = \frac{2\sqrt{r}}{1+r}\,, \quad 0 \le r \le 1$$

we obtain

$$1 \le \frac{\psi_{K,2^{n+1}}(x)}{\varphi_{K,2^{n+1}}(x)} = \frac{\Phi_2(\psi_{K,2^n}(\Phi_{1/2}(x)))}{\Phi_2(\varphi_{K,2^n}(\Phi_{1/2}(x)))} = \left(\frac{\psi_{K,2^n}(\Phi_{1/2}(x))}{\varphi_{K,2^n}(\Phi_{1/2}(x))}\right)^{1/2}$$
$$\times \left(\frac{1+\varphi_{K,2^n}(\Phi_{1/2}(x))}{1+\psi_{K,2^n}(\Phi_{1/2}(x))}\right) \le \sup_{0<x\le 1}\left(\frac{\psi_{K,2^n}(x)}{\varphi_{K,2^n}(x)}\right)^{1/2}.$$

Hence and by (1.10) we conclude that the inequality (1.8) holds. In a similar way we arrive at the inequality (1.9) which ends the proof.

As shown in [1]

(1.12) $$\Phi_{1/K} = h \circ \Phi_K \circ h\,, \quad K > 0,$$

so in view of the above theorem we easily obtain a result complementary to [8, Corollary 1.6]

Corollary 1.3. *For any* $0 \le x \le 1$ *and* $n = 0,1,2,\ldots$

$$0 \le \widetilde{\psi}_{K,2^n}(x) - \widetilde{\varphi}_{K,2^n}(x) \le 2(1 - 4^{(1-K)2^{-n}})h^K(x) \quad as \quad K \ge 1 \tag{1.13}$$

and

$$0 \le \widetilde{\varphi}_{K,2^n}(x) - \widetilde{\psi}_{K,2^n}(x) \le 2(4^{(1-K)2^{-n}} - 1)h^K(x) \quad as \quad 0 < K < 1 \tag{1.14}$$

Proof. Let $K \ge 1$ be any fixed number. By the inequality

$$|h(x) - h(y)| \le 2|x - y|, \quad 0 \le x, y \le 1,$$

the equalities (1.2), (1.3) and the estimate (1.9) we get for any $0 < x \le 1$

$$\begin{aligned}\widetilde{\psi}_{K,2^n}(x) - \widetilde{\varphi}_{K,2^n}(x) &= h \circ \varphi_{1/K,2^n} \circ h(x) - h \circ \psi_{1/K,2^n} \circ h(x)\\ &\le 2(\varphi_{1/K,2^n}(h(x)) - \psi_{1/K,2^n}(h(x)))\\ &\le 2(1 - 4^{(1-K)2^{-n}})\varphi_{1/K,2^n}(h(x)) \le 2(1 - 4^{(1-K)2^{-n}})h^K(x)\end{aligned}$$

because of Theorem 1.1. This proves the inequality (1.13). In a similar way we derive the estimate (1.14) and this ends the proof.

2. Estimates of the function μ^{-1}, λ, μ. In [8] we have given some estimates by elementary functions of the distortion function Φ_K as a consequence of [8, Theorem 1.3, Corollary 1.4]. For this cf. [8, Theorems 2.1, 2.2]. In this section we shall establish the estimates for the familiar functions μ^{-1}, λ, μ by elementary functions.* Since these functions can be expressed by means of the function Φ_K so we may apply Theorems 1.1, 1.2 and Corollary 1.3 to approximate them. Namely, it follows from the equality (0.2) that $\mu(1/\sqrt{2}) = \pi/2$. Hence and by (0.1)

$$\mu^{-1}\left(\frac{\pi}{2}r\right) = \Phi_{1/r}\left(\frac{1}{\sqrt{2}}\right), \quad r > 0. \tag{2.1}$$

Moreover, by an equality shown in [1]

$$\Phi_K^2(x) + \Phi_{1/K}^2(\sqrt{1 - x^2}) = 1 \tag{2.2}$$

and by the following formula, cf. [1],

$$\lambda(r) = \frac{\Phi_r^2(1/\sqrt{2})}{\Phi_{1/r}^2(1/\sqrt{2})} \tag{2.3}$$

we get

$$\left[\mu^{-1}\left(\frac{\pi}{2}\frac{1}{r}\right)\right]^2 + \left[\mu^{-1}\left(\frac{\pi}{2}r\right)\right]^2 = 1, \quad r > 0 \tag{2.4}$$

* μ^{-1} denotes here and further on the inverse function to μ.

and

$$\frac{1}{\sqrt{1+\lambda(r)}} = \Phi_{1/r}\Big(\frac{1}{\sqrt{2}}\Big) = \mu^{-1}\Big(\frac{\pi}{2}r\Big)\ , \quad r>0\,. \tag{2.5}$$

As a direct application of Theorem 1.1 to the equalities (2.1) and (2.5) we derive

Corollary 2.1. *For every $r \geq 1$ $(0 < r < 1)$ the functions $\psi_{1/r,t}(1/\sqrt{2})$, $\widetilde{\psi}_{1/r,t}(1/\sqrt{2})$ of a real parameter $t \geq 1$ are increasing (decreasing) whereas $\varphi_{1/r,t}(1/\sqrt{2})$, $\widetilde{\varphi}(1/\sqrt{2})$ are decreasing (increasing), respectively. Moreover,*

$$\mu^{-1}\Big(\frac{\pi}{2}r\Big) = \frac{1}{\sqrt{1+\lambda(r)}} = \lim_{t\to+\infty}\varphi_{1/r,t}\Big(\frac{1}{\sqrt{2}}\Big) = \lim_{t\to+\infty}\widetilde{\varphi}_{1/r,t}\Big(\frac{1}{\sqrt{2}}\Big) =$$
$$= \lim_{t\to+\infty}\psi_{1/r,t}\Big(\frac{1}{\sqrt{2}}\Big) = \lim_{t\to+\infty}\widetilde{\psi}_{1/r,t}\Big(\frac{1}{\sqrt{2}}\Big).$$

Setting $t = 2^n$, $n = 0,1,2,\ldots$ we derive in view of the above corollary the following estimates

$$\begin{aligned} &\mu^{-1}\Big(\frac{\pi}{2}r\Big) = \frac{1}{\sqrt{1+\lambda(r)}} \leq \min\Big\{\psi_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big),\widetilde{\psi}_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big\} \\ &\mu^{-1}\Big(\frac{\pi}{2}r\Big) \geq \max\Big\{\varphi_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big),\widetilde{\varphi}_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big\} \quad \text{as } 0<r\leq 1 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\mu^{-1}\Big(\frac{\pi}{2}r\Big) = \frac{1}{\sqrt{1+\lambda(r)}} \leq \min\Big\{\varphi_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big),\widetilde{\varphi}_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big\} \\ &\mu^{-1}\Big(\frac{\pi}{2}r\Big) \geq \max\Big\{\psi_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big),\widetilde{\psi}_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big\} \quad \text{as } r\geq 1, \end{aligned} \tag{2.7}$$

expressed thanks to (1.11) by elementary functions, and the accuracy grows step by step as n increases to infinity. In particular for $n = 2$ we get

Theorem 2.2. *If $0 < r \leq 1$ then*

$$f(p^{1/r}) \leq \mu^{-1}\Big(\frac{\pi}{2}r\Big) \leq f(4^{1-1/r}p^{1/r}) \tag{2.8}$$

where $f(x) = h \circ \Phi_4(x) = (\sqrt{1+x} - \sqrt[4]{4x})^2(\sqrt{1+x} + \sqrt[4]{4x})^{-2}$, $0 \leq x \leq 1$ and $p = \Phi_{1/4} \circ h(1/\sqrt{2}) \approx 0.0000139494$. If $r \geq 1$ then

$$\sqrt{1-f^2(4^{1-r}p^r)} = \Phi_8(4^{1-r}p^r) \leq \mu^{-1}\Big(\frac{\pi}{2}r\Big) \leq \sqrt{1-f^2(p^r)} = \Phi_8(p^r)\ . \tag{2.9}$$

Proof. Setting $n = 2$ we conclude by the inequality (2.6) that for $0 < r \leq 1$

$$\widetilde{\varphi}_{1/r,4}\Big(\frac{1}{\sqrt{2}}\Big) \leq \mu^{-1}\Big(\frac{\pi}{2}r\Big) \leq \widetilde{\varphi}_{1/r,4}\Big(\frac{1}{\sqrt{2}}\Big).$$

But $\widetilde{\varphi}_{1/r,4}(x) = f((f(x))^{1/r})$ and $\widetilde{\psi}_{1/r,4}(x) = f(4^{1-1/r}(f(x))^{1/r})$ because of (1.2), (1.3) and (1.12) which proves the inequality (2.8). The next bounds (2.9) immediately follow from (2.8) and (2.4).

For r close to 1 the bounds (2.8) and (2.9) are quite precise, see Corollary 3.2. The equalities (2.5), (1.11) and the inequality (2.9) lead directly to

Corollary 2.3. *For any $r \geq 1$ the following inequalities hold*

$$\lambda(r) \leq \frac{1}{4}\Big(\Phi_4(4^{1-r}p^r) + \frac{1}{\Phi_4(4^{1-r}p^r)}\Big) - \frac{1}{2} = \frac{1}{\Phi_8^2(4^{1-r}p^r)} - 1 \tag{2.10}$$

$$\lambda(r) \geq \frac{1}{4}\Big(\Phi_4(p^r) + \frac{1}{\Phi_4(p^r)}\Big) - \frac{1}{2} = \frac{1}{\Phi_8^2(p^r)} - 1\,. \tag{2.11}$$

From the above inequality we shall derive a slightly more convenient estimate.

Corollary 2.4. *For any $r \geq 1$ the following inequalities hold*

$$\lambda(r) - 1 \leq A\Big(\Big(\frac{2}{\sqrt{p}}\Big)^{r-1} - 1\Big) - B\Big(1 - \Big(\frac{\sqrt{p}}{2}\Big)^{r-1}\Big) \tag{2.12}$$

where

$$A = \frac{p^{-1/2}}{32}\Phi_2^{1/2}(p) \approx 0.723142$$

and

$$B = p^{1/2}\left(\frac{1}{2}\frac{1-\Phi_2(p)}{(1+\Phi_2(p))^2\Phi_2^{1/2}(p)}\frac{1-p}{(1+p)^2} + \frac{1}{8}\frac{1-p}{(1+p)^2\Phi_2^{1/2}(p)} + \frac{1}{32}\Phi_2^{1/2}(p)\right)$$
$$\approx 0.0265396\,.$$

On the other hand

$$\lambda(r) - 1 \geq \frac{2\Phi_8'(p)}{\Phi_8^3(p)}(p - p^r)\,, \quad \frac{2\Phi_8'(p)}{\Phi_8^3(p)} \approx 24968.9 \tag{2.13}$$

where p is the constant from Theorem 2.2.

Proof. Assume that a, b are arbitrary numbers such that $0 < a \leq b \leq 1$. It follows from (1.11) that

$$\Phi_2(b) - \Phi_2(a) \geq 2\frac{1-b}{(1+b)^2}(\sqrt{b} - \sqrt{a}) \geq \frac{1-b}{(1+b)^2\sqrt{b}}(b-a)\,.$$

Hence

$$\Phi_4(b) - \Phi_4(a) \geq 2\frac{1-\Phi_2(b)}{(1+\Phi_2(b))^2\Phi_2^{1/2}(b)}\frac{1-b}{(1+b)^2}(\sqrt{b} - \sqrt{a}) \tag{2.14}$$

and

$$\Phi_2^{1/2}(b) - \Phi_2^{1/2}(a) \geq \frac{\Phi_2(b) - \Phi_2(a)}{2\Phi_2^{1/2}(b)} \geq \frac{1-b}{(1+b)^2\Phi_2^{1/2}(b)}(\sqrt{b} - \sqrt{a}) . \tag{2.15}$$

Moreover, in view of (1.11), we get
(2.16)

$$\Phi_2^{-1/2}(a) - \Phi_2^{-1/2}(b) \leq \frac{\Phi_2^{-1}(a) - \Phi_2^{-1}(b)}{\Phi_2^{-1/2}(a) + \Phi_2^{-1/2}(b)} \leq \frac{1}{4}\Phi_2^{1/2}(b)\Big(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} + \sqrt{a} - \sqrt{b}\Big) .$$

Setting $b = p$, $a = 4^{1-r}p^r$, $r \geq 1$ we obtain by virtue of Corollary 2.3 (2.10) and (1.11) that

$$\begin{aligned}\lambda(r) - 1 &\leq \frac{1}{4}\Big(\Phi_4(a) - \Phi_4(b) + \frac{1}{\Phi_4(a)} - \frac{1}{\Phi_4(b)}\Big) \\ &= \frac{1}{4}(\Phi_4(a) - \Phi_4(b)) + \frac{1}{8}(\Phi_2^{1/2}(a) - \Phi_2^{1/2}(b)) + \frac{1}{8}(\Phi_2^{-1/2}(a) - \Phi_2^{-1/2}(b)) .\end{aligned}$$

This and the equalities (2.14), (2.15), (2.16) yield the equality (2.12).

On the other hand, it follows from (1.11) that the function Φ_2 is concave. Hence the function Φ_8 is concave as well, see also [1], which yields the convexity of the function Φ_8^{-2}. This and the inequality (2.11) from Corollary 2.3 prove the inequality (2.13) which ends the proof.

Now, we shall estimate the module function μ defined by the equality (0.2). To this end we shall apply the asymptotic behaviour of μ near 0 given by

$$\lim_{r \to 0^+} (\mu(r) + \log r) = \log 4 , \tag{2.17}$$

cf. [6]. From (2.17) and by the equality

$$\mu(\Phi_{1/K}(r)) = \mu(\mu^{-1}(K\mu(r))) = K\mu(r)$$

we have for any r, $0 < r < 1$

$$\begin{aligned}0 &= \lim_{K \to +\infty} \frac{1}{K}(\mu(\Phi_{1/K})) + \log \Phi_{1/K}(r)) = \\ &= \lim_{K \to +\infty} \big(\mu(r) + \frac{1}{K}\log \Phi_{1/K}(r)\big) = \mu(r) + \lim_{K \to +\infty} \frac{1}{K}\log \Phi_{1/K}(r) .\end{aligned}$$

Thus

$$\mu(r) = -\lim_{K \to +\infty} \frac{1}{K}\log \Phi_{1/K}(r) , \quad 0 < r < 1 . \tag{2.18}$$

Suppose ψ_t and φ_t, $0 < t \leq 1$, are arbitrary mappings of the interval $[0,1]$ into itself such that

$$\psi_t(r) \leq \Phi_t(r) \leq \varphi_t(r) , \quad 0 < t \leq 1,\ 0 \leq r \leq 1 .$$

Hence for every $n = 0, 1, 2, \dots$

$$\text{(2.19)}\quad \Phi_{2^n} \circ \psi_t \circ \Phi_{1/2^n}(r) \le \Phi_t(r) \le \Phi_{2^n} \circ \varphi_t \circ \Phi_{1/2^n}(r)\,, \quad 0 < t \le 1,\ 0 \le r \le 1\,.$$

Then keeping n fixed we derive from the inequality (2.19) and the equality (2.18) that for any $0 < r < 1$

$$\begin{aligned}\mu(r) &\ge -\lim_{K\to+\infty} \frac{1}{K}\log \Phi_{2^n}(\varphi_{1/K} \circ \Phi_{1/2^n}(r))\\ &= -\lim_{K\to+\infty} \frac{1}{K}\log \frac{2(\Phi_{2^{n-1}}(\varphi_{1/K} \circ \Phi_{1/2^n}(r)))^{1/2}}{1 + \Phi_{2^{n-1}}(\varphi_{1/K} \circ \Phi_{1/2^n}(r))}\\ &= -\lim_{K\to+\infty} \frac{1}{2K}\log \Phi_{2^{n-1}}(\varphi_{1/K} \circ \Phi_{1/2^n}(r))\\ &= \dots = -\lim_{K\to+\infty} \frac{1}{2^n K}\log \varphi_{1/K} \circ \Phi_{1/2^n}(r)\,.\end{aligned}$$

Similarly

$$\mu(r) \le -\lim_{K\to+\infty} \frac{1}{2^n K}\log \psi_{1/K} \circ \Phi_{1/2^n}(r)\,.$$

Thus setting for any fixed $m = 1, 2, \dots$

$$\varphi_t(r) = (1 - r^{2^{m+1}})^{-1/t2^m}\psi_{t,2^m}(r) \qquad \psi_t(r) = \psi_{t,2^m}(r)$$

we obtain by virtue of Theorem 1.1 and (1.1), (1.2) the following bounds

$$\begin{aligned}\mu(r) &\le -\lim_{K\to+\infty} \frac{1}{2^n K}\log \psi_{1/K,2^m} \circ \Phi_{1/2^n}(r)\\ &= -\lim_{K\to+\infty} \frac{1}{2^n K}\log \Phi_{2^m} \circ \psi_{1/K,1} \circ \Phi_{1/2^m} \circ \Phi_{1/2^n}(r)\\ &= -\lim_{K\to+\infty} \frac{1}{2^n K}\log \Phi_{2^m}(\psi_{1/K,1} \circ \Phi_{1/2^{m+n}}(r))\\ &= -\lim_{K\to+\infty} \frac{1}{2^n K}\log \frac{2(\Phi_{2^{m-1}}(\psi_{1/K,1} \circ \Phi_{1/2^{m+n}}(r)))^{1/2}}{1 + \Phi_{2^{m-1}}(\psi_{1/K,1} \circ \Phi_{1/2^{m+n}}(r))}\\ &= -\lim_{K\to+\infty} \frac{1}{2^{n+1} K}\log \Phi_{2^{m-1}}(\psi_{1/K,1} \circ \Phi_{1/2^{m+n}}(r))\\ &= \dots = -\lim_{K\to+\infty} \frac{1}{2^{n+m} K}\log 4^{1-K}\Phi^K_{1/2^{m+n}}(r) = \frac{1}{2^{n+m}}\log 4\Phi^{-1}_{1/2^{m+n}}(r)\end{aligned}$$

and hence

$$\begin{aligned}\mu(r) &\ge -\lim_{K\to+\infty} \frac{1}{2^n K}\log \varphi_{1/K} \circ \Phi_{1/2^n}(r)\\ &= -\lim_{K\to+\infty} \frac{1}{2^n K}\log(1 - (\Phi_{1/2^n}(r))^{2^{m+1}})^{-K/2^m}\\ &\quad -\lim_{K\to+\infty} \frac{1}{2^n K}\log \psi_{1/K,2^m} \circ \Phi_{1/2^n}(r)\\ &= \frac{1}{2^{n+m}}\log(1 - (\Phi_{1/2^n}(r))^{2^{m+1}}) + \frac{1}{2^{n+m}}\log 4\Phi^{-1}_{1/2^{m+n}}(r)\,.\end{aligned}$$

This way we have proved the following

Theorem 2.5. *For any $n = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$ and $0 < r < 1$ the following bounds hold*

$$0 \le \frac{1}{2^{n+m}} \log 4\Phi^{-1}_{1/2^{m+n}}(r) - \mu(r) \le -\frac{1}{2^{n+m}} \log(1 - (\Phi_{1/2^n}(r))^{2^{m+1}}).$$

The above estimates are improvements of the classical bounds for the function μ given in [6]. They improve as well recent results of Vuorinen, Vamanamurthy and Anderson, cf. [1], [3], [9], [10]. Setting, for example, $n = 0, 1$ and $m = 1$ we obtain by Theorem 2.5 the following inequalities

$$0 \le \log \frac{2(1+\sqrt{1-r^2})}{r} - \mu(r) \le -\frac{1}{2}\log(1-r^4)$$

and

$$0 \le \frac{1}{2}\log \frac{2(1+\sqrt{1-r^2})(1+\sqrt[4]{1-r^2})^2}{r^2} - \mu(r) \le -\frac{1}{4}\log\left(1 - \frac{r^8}{(1+\sqrt{1-r^2})^8}\right),$$

respectively, where r ranges from 0 to 1.

3. Numerical applications.

The functions $\varphi_{K,2^n}$, $\psi_{K,2^n}$, $\widetilde{\varphi}_{K,2^n}$, $\widetilde{\psi}_{K,2^n}$, $n = 0, 1, 2, \ldots$, $K > 0$, can be applied for numerical calculations of the function Φ_K (and also of μ^{-1} and λ because of (2.5)) with an arbitrarily preassigned accuracy. In this section we shall estimate the error of the approximation as an application of Theorems 1.1, 1.2 and Corollary 1.3.

Theorem 3.1. *If $K \ge 1$ and p is an arbitrarily fixed number such that $h(\sqrt{0.1}) \le p \le \Phi_2(0.1)$ then*

$$(3.1)\quad \begin{aligned} 0 &\le \psi_{K,2^n}(x) - \Phi_K(x) \\ &\le \min\left\{\left(\frac{x}{1+\sqrt{1-x^2}}\right)^{2^{n+1}/K}, 4^{1-1/K}(4^{(1-1/K)2^{-n}} - 1)x^{1/K}\right\} < (0.1)^{2^n/K} \end{aligned}$$

as $0 \le x \le p$ and $n = 2, 3, \ldots$ but

$$(3.2)\quad 0 \le \widetilde{\psi}_{K,2^n} - \Phi_K(x) \le \min\left\{3h(x)^{2^{n+1}}, 2(1 - 4^{(1-K)2^{-n}})h^K(x)\right\} < 3\cdot(0.1)^{2^{n+1}}$$

as $p \le x \le 1$ and $n = 1, 2, 3, \ldots$

Proof. By the equality (1.11) we have

$$4\Phi_{1/2t}(x) = 4\Phi_{1/2}\circ\Phi_{1/t}(x) \ge \Phi^2_{1/t}(x)\,,\quad t \ge 1\,,\ 0 \le x \le 1\,.$$

from which

$$\psi_{2K,1}\circ\Phi_{1/2t}(x)=\min\{4^{1-1/2K}\Phi_{1/2t}^{1/2K}(x),1\}\geq\min\{4^{1-1/K}\Phi_{1/t}^{1/K}(x),1\}\geq$$
$$\geq\psi_{K,1}\circ\Phi_{1/t}(x)\,,\quad t\geq 1,\ 0\leq x\leq 1\,.$$

Hence for every $t, K\geq 1$ and $0\leq x\leq 1$

$$\psi_{K,t}(x)=\Phi_t\circ\psi_{K,1}\circ\Phi_{1/t}(x)\leq\Phi_t\circ\psi_{2K,1}\circ\Phi_{1/2t}(x)=\psi_{2K,t}(\Phi_{1/2}(x))\,.$$

Then, applying (1.4) we get for any $n=2,3,\ldots$

$$0\leq\psi_{K,2^n}(x)-\Phi_K(x)\leq\psi_{2K,2^n}(\Phi_{1/2}(x))-\Phi_{2K}(\Phi_{1/2}(x))$$
$$\leq(\Phi_{1/2}(x))^{2^{n+1}/2K}\psi_{2K,2^n}(\Phi_{1/2}(x))\leq(\Phi_{1/2}(x))^{2^n/K}<(0.1)^{2^n/K}\quad\text{as } 0\leq x\leq p\,.$$

On the other hand, taking into account (1.8), we obtain the inequality (3.1). Now, we prove the second inequality. Let $q=2^{-n}K$ where $K>0$, $n=1,2,\ldots$ are arbitrarily fixed. Setting $r(q,v)=2((1-v)^{-q}-1)v^{q/2}$ we get by (1.5)

$$\tag{3.3} 0\leq\widetilde{\psi}_{K,2^n}(x)-\Phi_K(x)\leq r(q,h(x)^{2^{n+1}})\,,\quad 0<x<1\,.$$

If $0<q\leq 1$ then

$$\tag{3.4} r(q,v)\leq 2\Big(\frac{1}{1-v}-1\Big)v^{q/2}=\frac{2v^{q/2}}{1-v}v\leq 3v \text{ as } v\leq 0.1.$$

If $1<q\leq 2$ then

$$\tag{3.5} r(q,v)\leq 2\Big(\frac{1}{(1-v)^2}-1\Big)v^{q/2}\leq\frac{2v^{1/2}(2-v)}{(1-v)^2}v\leq 2v \text{ as } v\leq 0.1.$$

If $q>2$ then

$$\tag{3.6} r(q,v)\leq 2\frac{v^{q/2}}{(1-v)^q}=\frac{2}{(1-v)^2}\Big(\frac{v^{1/2}}{1-v}\Big)^{q-2}\cdot v\leq\frac{2}{(1-v)^2}v\leq 3v \text{ as } v\leq 0.1.$$

All these four inequalities (3.3)–(3.6) give after a substitution $v=h(x)^{2^{n+1}}$ the following estimate

$$0\leq\widetilde{\psi}_{K,2^n}(x)-\Phi_K(x)\leq 3h(x)^{2^{n+1}}\quad\text{as } p\leq x\leq 1.$$

This together with (1.13) proves the inequality (3.2) and ends the proof.

From the above theorem and the equalities (2.1), (2.4) we derive

Corollary 3.2. *If $0<r\leq 1$ then for every $n=1,2,3,\ldots$*

$$\tag{3.7} 0\leq\widetilde{\psi}_{1/r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)-\mu^{-1}\Big(\frac{\pi}{2}r\Big)$$
$$\leq\min\{3(\sqrt{2}-1)^{2^{n+2}},2(1-4^{(1-1/r)2^{-n}})(\sqrt{2}-1)^{2/r}\}\leq 3\cdot(\sqrt{2}-1)^{2^{n+2}}$$

whereas for $r \geq 1$

$$(3.8)\quad 0 \leq \mu^{-1}\Big(\frac{\pi}{2}r\Big) - \Big(1 - \widetilde{\psi}^2_{r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big)^{1/2} \leq 2^{r-1/2}\min\{3(\sqrt{2}-1)^{2^{n+3}-r+1}, 2(1-4^{(1-r)2^{-n}})(\sqrt{2}-1)^{r+1}\}.$$

Proof. The inequality (3.7) is an immediate consequence of (3.2) and the equality (2.1). Assume that $r \geq 1$ is arbitrary. By (1.2), (1.3) and Theorem 1.1 we obtain for any $n = 1, 2, \ldots$

$$(3.9)\quad \frac{1}{1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2})} = \frac{1}{h \circ \widetilde{\psi}_{r,2^n}(1/\sqrt{2})}\,\frac{1}{(1+\widetilde{\psi}_{r,2^n}(1/\sqrt{2}))^2} \leq \frac{1}{\psi_{1/r,2^n}(h(1/\sqrt{2}))}\cdot\frac{1}{(1+1/\sqrt{2})^2} \leq \frac{1}{2}\,4^r(\sqrt{2}-1)^{2-2r}.$$

Since, in view of (2.4) and (3.7),

$$(3.10)\quad 0 \leq \mu^{-1}\Big(\frac{\pi}{2}r\Big) - \Big(1-\widetilde{\psi}^2_{r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big)^{1/2} = \frac{\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2}) - (\mu^{-1}(\pi/2r))^2}{\sqrt{1-(\mu^{-1}(\pi/2r))^2} + \sqrt{1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2})}} \leq \frac{\psi_{r,2^n}(1/\sqrt{2}) - \mu^{-1}(\pi/2r)}{\sqrt{1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2})}},$$

$n = 1, 2, \ldots$, so applying the inequalities (3.7) and (3.9) we obtain the inequality (3.8) which ends the proof.

The equality (2.3) implies $\lambda(1/r) = 1/\lambda(r)$, $r > 0$, so in the following corollary we may restrict ourselves to the case $r \geq 1$.

Corollary 3.3. *For any* $r \geq 1$ *and* $n = 1, 2, 3, \ldots$

$$(3.11)\quad 0 \leq \Big(1-\widetilde{\psi}^2_{r,2^n}\Big(\frac{1}{\sqrt{2}}\Big)\Big)^{-1} - 1 - \lambda(r) \leq 2\cdot 4^r e^{\pi(r-1/r)}\min\{3(\sqrt{2}-1)^{2^{n+2}+2-2r}, 2(\sqrt{2}-1)^2(1-4^{(1-r)2^{-n}})\}.$$

Proof. It follows from (2.2) and (2.3) that

$$(3.12)\qquad \lambda(r) = \frac{\Phi^2_r(1/\sqrt{2})}{\Phi^2_{1/r}(1/\sqrt{2})} = \frac{\Phi^2_r(1/\sqrt{2})}{1-\Phi^2_r(1/\sqrt{2})} = \frac{1}{1-\Phi^2_r(1/\sqrt{2})} - 1, \quad r > 0.$$

This and (3.2) imply

$$(3.13)\quad 0 \leq \frac{1}{1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2})} - 1 - \lambda(r) = \frac{\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2}) - \Phi^2_r(1/\sqrt{2})}{(1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2}))(1-\Phi^2_r(1/\sqrt{2}))} \leq 4\,\frac{\widetilde{\psi}_{r,2^n}(1/\sqrt{2}) - \Phi_r(1/\sqrt{2})}{1-\widetilde{\psi}^2_{r,2^n}(1/\sqrt{2})}\,\lambda(r).$$

Since $\lambda(r) \le e^{\pi(r-1/r)}$, cf. [1], we derive by (3.7) and (3.9) the inequality (3.11), which ends the proof.

Corollary 3.3 shows that the above considered approximation of the function λ is good rather for small r, i.e. $1 \le r < 10$. Therefore the estimate (3.11) completes, in the case of small r, the following estimate, cf. [1],

$$\lambda(r) \le \frac{1}{16}e^{\pi r} - \frac{1}{2} + \delta(r)$$

where $e^{-\pi r} < \delta(r) < 2e^{-\pi r}$.

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STRESZCZENIE

Celem pracy jest podanie szeregu zastosowań metody aproksymacji funkcji dystorsji Φ_K. W rezultacie uzyskano nowe oszacowania funkcji μ^{-1}, λ, μ. Ponadto podano błąd wspomnianej wyżej aproksymacji, użyteczny w obliczeniach numerycznych.

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