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On the Strong Law of Large Numbers in Banach Space of Type p

O mocnym prawie wielkich liczb w przestrzeni Banacha typu p

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent random elements with values in a Banach space $\mathcal{X}$. It is shown that Teicher's version of the strong law of large numbers holds if and only if $\mathcal{X}$ is of type p. We extend results of [3] where Chung's version of SLLN is considered.

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a separable Banach space $(\mathcal{X}, \|\cdot\|)$. Assume that $\{\varepsilon_n, n \geq 1\}$ is a sequence of independent identically distributed Rademacher random variables, i.e. $P[\varepsilon_n = -1] = P[\varepsilon_n = 1] = 1/2$, $n \geq 1$.

A real separable Banach space is called of type p iff there exists a constant $A \in \mathbf{R}^+$ and that

$$(1.1) \qquad E\Big\|\sum_{n=1}^{\infty} \varepsilon_n x_n\Big\|^p \leq A \sum_{n=1}^{\infty} \|x_n\|^p$$

for each $\{x_n, n \geq 1\} \in \mathcal{X}^\infty = \prod_{n=1}^{\infty} \mathcal{X}$.

We say that a sequence $\{X_n, n \geq 1\}$ of random elements with $EX_n = 0$, $n \geq 1$, taking values in a Banach space satisfies the strong law of large numbers (abbr. SLLN) if

$$(1.2) \qquad \|S_n\|/n \to 0 \text{ , a.s. } , n \to \infty \text{ , where } S_n = \sum_{j=1}^{n} X_j \text{ .}$$

In [3] it is shown that Chung's version of SLLN holds, if and only if $\mathcal{X}$ is of type p. The aim of this note is to give new conditions characterizing Banach spaces of type p in terms of the validity of SLLN.

2. The strong law of large numbers. We shall use the following lemmas.

Lemma 2.1. ([6]) *Let $X_1,\dots,X_n$ be independent $\mathcal{X}$-valued random variables with $E\|X_i\|<\infty$ $(i=1,\dots,n)$. Let $\mathcal{F}_k$ be the σ-field generated by $(X_1,\dots,X_k)$ $(k=1,\dots,n)$ and let $\mathcal{F}_0$ be the trivial σ-field. Then for $1\le k\le n$,*

$$\left|E(\|S_n\|\,|\mathcal{F}_k)-E(\|S_n\|\,|\mathcal{F}_{k-1})\right|\le\|X_k\|+E\|X_k\|\,.$$

Lemma 2.2. ([1]) *Let $X_1,\dots,X_n$ be independent $\mathcal{X}$-valued random variables. Then*

$$S_n/n\to0 \text{ a.s. } \textit{ iff } S_{2^k}/2^k\to0 \text{ a.s. } \textit{ and } S_n/n\to0 \textit{ in probability.}$$

The following result generalizes Theorem 2.1. of [3].

Theorem 2.3. *Let $1\le p\le2$, then the following statements are equivalent:*

(i) *$\mathcal{X}$ is a Banach space of type p.*

(ii) *There exists a constant $A\in\mathbf{R}^+$ such that*

$$E\Big\|\sum_{k=1}^n X_k\Big\|^p\le A\sum_{k=1}^n E\|X_k\|^p \tag{2.1}$$

for all independent random elements $X_1,\ \dots,\ X_n$ such that $EX_i=0$ and $E\|X_i\|^p<\infty$, $1\le i\le n$.

(iii)

$$n^{-1}\Big\|\sum_{i=1}^n X_i\Big\|\to0 \text{ a.s. },\ n\to\infty\ , \textit{ for all sequences } \{X_n,n\ge1\} \tag{2.2}$$

of independent random elements with $EX_i=0$, $E\|X_i\|^p<\infty$, $i\ge1$ and such that

(A) $\sum\limits_{j=2}^{\infty} j^{-2p}E\|X_j\|^p\sum\limits_{i=1}^{j-1}E\|X_i\|^p<\infty$,

(B) $n^{-p}\sum\limits_{i=1}^{n}E\|X_i\|^p=o(1)$,

(C) $\sum\limits_{n=1}^{\infty}P\big[\|X_n\|\ge\alpha_n\big]<\infty$,

for some sequence $\{\alpha_n,n\ge1\}$ of positive numbers such that

(D) $\sum\limits_{n=1}^{\infty}n^{-2p}\alpha_n^pE\|X_n\|^p<\infty$.

(iv) *Assume that*

(A_1) $\sum\limits_{j=2}^{\infty}j^{-2p}\|x_j\|^p\sum\limits_{i=1}^{j-1}\|x_i\|^p<\infty$,

(B_1) $n^{-p}\sum\limits_{i=1}^{n}\|x_i\|^p=o(1)$,

(C_1) *there exists a sequence $\{\alpha_n,n\ge1\}$ of positive numbers such that $\|x_n\|\ge\alpha_n$ holds only for finite many n and*

(D₁) $\sum_{n=1}^{\infty} n^{-2p}\alpha_n^p\|x_n\|^p < \infty$.

If $\{\varepsilon_n, n 1\}$ *is a Rademacher sequence of random variables, then*

$$n^{-1}\sum_{j=1}^{n}\varepsilon_j x_j \to 0 \text{ in probability },\ n\to\infty .$$

(v) $n^{-1}\left\|\sum_{i=1}^{n} X_i\right\| \to 0$ a.s. , $n\to\infty$, *for all sequences* $\{X_n, n\ge 1\}$ *of independent random elements with* $EX_i=0$, $E\|X_i\|^p<\infty$, $i\ge 1$ *and such that* $\sum_{j=1}^{\infty} j^{-p}E\|X_j\|^p<\infty$.

(vi) *If* $\sum_{j=1}^{\infty} j^{-p}\|x_j\|^p<\infty$, *and* $\{\varepsilon_n, n\ge 1\}$ *is a Rademacher sequence of random variables, then*

$$n^{-1}\sum_{j=1}^{n}\varepsilon_j x_j \to 0 \text{ in probability },\ n\to\infty .$$

(vii) $n^{-1}\left\|\sum_{j=1}^{n} X_j\right\| \to 0$ a.s. , $n\to\infty$, *for all sequences of* $\{X_n, n\ge 1\}$ *of independent symmetric random elements such that*

(A₂) $\sum_{j=2}^{\infty} j^{-2p}\|X_j\|^p\sum_{i=1}^{j-1}\|X_i\|^p<\infty$ a.s. ,

(B₂) $n^{-p}\sum_{i=1}^{n}\|X_i\|^p = o(1)$ a.s. ,

(C₂) $\sum_{n=1}^{\infty} P[\|X_n\|\ge\alpha_n]<\infty$,

for some sequence $\{\alpha_n, n\ge 1\}$ *of positive numbers such that*

(D₂) $\sum_{n=1}^{\infty} n^{-2p}\alpha_n^p\|X_n\|^p<\infty$ a.s. .

Proof. The equivalences

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(v)} \Leftrightarrow \text{(vi)}$$

have been proved in [3].

We have only to show the following implications

$$\begin{array}{ccccc} & & \text{(vii)} & & \\ \Uparrow & & & & \Downarrow \\ \text{(ii)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(iv)} \\ & & & & \Downarrow \\ & & & & \text{(vi)} \end{array}$$

(ii) $\Rightarrow$ (iii). Let $\{X_n, n\ge 1\}$ be any sequence of independent $\mathcal{X}$-valued random elements with $EX_i = 0$, $E\|X_i\|^p<\infty$, such that (2.1) holds, and assume that the conditions (A)–(D) of (iii) are satisfied.

Putting $X'_k = X_k I[\|X_k\| < k]$, we see that

$$(2.3)\quad P\Big[n^{-1}\Big\|\sum_{k=1}^{n}(X'_k - EX'_k)\Big\| \geq \varepsilon\Big] \leq n^{-p}\varepsilon^{-p}E\Big\|\sum_{k=1}^{n}(X'_k - EX'_k)\Big\|^p$$

$$\leq An^{-p}\varepsilon^{-p}\sum_{k=1}^{n}E\|X'_k - EX'_k\|^p$$

$$\leq 2^{p-1}An^{-p}\varepsilon^{-p}\sum_{k=1}^{n}E\|X'_k\|^p = o(1).$$

Write

$$X^*_k = X'_k - EX'_k, \quad k \geq 1, \quad S'_n = \sum_{k=1}^{n}X'_k, \quad S^*_n = \sum_{k=1}^{n}X^*_k,$$

and define

$$Y_{n,i} = E(\|S^*_n\| \,|\mathcal{F}^*_i) - E(\|S^*_n\| \,|\mathcal{F}^*_{i-1}),$$

where $\mathcal{F}^*_0 = \{\emptyset, \Omega\}$, $\mathcal{F}^*_i = \sigma(X^*_1, \ldots, X^*_i)$. Then we have

$$\|S^*_n\| - E\|S^*_n\| = \sum_{i=1}^{n}Y_{n,i}.$$

We prove now that $E\|S^*_n\| \to 0$, $n \to \infty$. By Lemma 2.1 we have

$$(2.4)\quad P\Big[\big|\,\|S^*_n\| - E\|S^*_n\|\,\big| > n\varepsilon\Big] \leq n^{-2}\varepsilon^{-2}E\Big(\sum_{i=1}^{n}Y_{n,i}\Big)^2$$

$$= n^{-2}\varepsilon^{-2}\sum_{i=1}^{n}EY^2_{n,i} \leq n^{-2}\varepsilon^{-2}\sum_{i=1}^{n}E(\|X^*_i\| + E\|X^*_i\|)^2$$

$$\leq 8n^{-2}\varepsilon^{-2}\sum_{i=1}^{n}E\|X'_i\|^2 \leq 8n^{-p}\varepsilon^{-2}\sum_{i=1}^{n}E\|X_i\|^p = o(1),$$

after using (B). Therefore, taking into account (2.3) we conclude that $E\|S^*_n\|/n \to 0$, $n \to \infty$.

To prove that $\|S^*_n\|/n \to 0$ a.s., $n \to \infty$, it is enough to see that $\|S^*_k\|/2^k \to 0$ a.s., $k \to \infty$ (cf. Lemma 2.2), or equivalently that $(\|S^*_{2^k}\| - E\|S^*_{2^k}\|)/2^k \to 0$ a.s., $k \to \infty$, as $E(\|S^*_{2^k}\|)/2^k \to 0$, $k \to \infty$.

Note that

$$(2.5)\quad (\|S^*_{2^n}\| - E\|S^*_{2^n}\|)^2/2n = 2^{-2^n}\sum_{i=1}^{2^n}Y^2_{2^n,i} + 2^{-2^n}\sum_{i=2}^{2^n}Y_{2^n,i}\sum_{j=1}^{i-1}Y_{2^n,j}.$$

Now put

$$Z_{2^n,i} = Y^2_{2^n,i}I[\|X_i\| < \alpha_i] - E(Y^2_{2^n,i}I[\|X_i\| < \alpha_i]\,|\mathcal{F}^*_{i-1}), \quad 1 \leq i \leq 2^n.$$

Then, by Lemma 2.1, and the assumptions (C)-(D) we have

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right| \geq 2^{2n}\varepsilon\right] \leq \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n} E\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right|^2$$

$$= \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} EZ_{2^n,i}^2 \leq \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} EY_{2^n,i}^4 I[\|X_i\| < \alpha_i]$$

$$\leq \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} E(\|X_i^*\| + E\|X_i^*\|)^4 I[\|X_i\| < \alpha_i]$$

$$\leq \varepsilon^{-2}2^8\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} E\|X_i'\|^4 I[\|X_i\| < \alpha_i]$$

$$\leq \varepsilon^{-2}(2^{12}/15)\sum_{i=1}^{\infty} i^{-4} E\|X_i'\|^4 I[\|X_i\| < \alpha_i]$$

$$\leq \varepsilon^{-2}(2^{12}/15)\sum_{i=1}^{\infty} i^{-2p} E\|X_i'\|^{2p} I[\|X_i\| < \alpha_i]$$

$$\leq \varepsilon^{-2}(2^{12}/15)\sum_{i=1}^{\infty} i^{-2p}\alpha_i^p E\|X_i\|^p < \infty\,.$$

Hence, by Borel-Cantelli lemma, we get

$$2^{-2n}\left\{\sum_{i=1}^{2^n} Y_{2^n,i}^2 I[\|X_i\| < \alpha_i] - \sum_{i=1}^{2^n} E(Y_{2^n,i}^2 I[\|X_i\| < \alpha_i]|\mathcal{F}_{i-1}^*)\right\} \to 0 \text{ a.s. }, n \to \infty\,.$$

Moreover, we see that (B) implies

$$2^{-2n}\sum_{i=1}^{2^n} E(Y_{2^n,i}^2 I[\|X_i\| < \alpha_i]\,|\mathcal{F}_{i-1}^*) \leq 2^{-2n}\sum_{i=1}^{2^n} E(Y_{2^n,i}^2|\mathcal{F}_{i-1}^*)$$

$$= 2^{-2n}\sum_{i=1}^{2^n} E(\|X_i^*\| + E\|X_i^*\|)^2 \leq 8\cdot 2^{-2n}\sum_{i=1}^{2^n} E\|X_i'\|^2$$

$$\leq 16\cdot(2^{-2n})^{-p}\sum_{i=1}^{2^n} E\|X_i'\|^p = o(1)\,.$$

Therefore, we obtain

$$2^{-2n}\sum_{i=1}^{2^n} Y_{2^n,i}^2 \to 0 \text{ a.s. }, n \to \infty\,.$$

Now, note that $\{Y_{n,i}\sum_{j=1}^{i-1} Y_{n,j},\ 1 \leq i \leq n\}$ and $\{Y_{n,i},\ 1 \leq i \leq n\}$ are martingale difference sequences for fixed n.

Hence, using Lemma 2.1 we get

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{i=2}^{2^n}\sum_{j=1}^{i-1} Y_{2^n,i}Y_{2n,j}\right| \geq 2^{2n}\varepsilon\right] \leq \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E\Big(Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j}\Big)^2$$

$$\leq \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E\Big\{(\|X_i^*\| + E\|X_i^*\|)^2\Big(\sum_{j=1}^{i-1} Y_{2^n,j}\Big)^2\Big\}$$

$$= \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E(\|X_i^*\| + E\|X_i^*\|)^2\sum_{j=1}^{i-1} EY_{2^n,j}^2$$

$$\leq \varepsilon^{-2}2^8\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E\|X_i'\|^2\sum_{j=1}^{i-1} E\|X_j'\|^2$$

$$\leq \varepsilon^{-2}(2^{12}/15)\sum_{i=2}^{\infty} i^{-2p}E\|X_i'\|^p\sum_{j=1}^{i-1} E\|X_j'\|^p < \infty .$$

Therefore, by Borel-Cantelli lemma we get

$$2^{-2n}\sum_{i=2}^{2^n} Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j} \to 0 \text{ a.s. },\ n \to \infty .$$

Thus, $n^{-1}\|\sum_{j=1}^{n} X_j' - EX_j'\| \to 0$ a.s., $n \to \infty$. Taking into account that

$$\sum_{k=1}^{\infty} P[X_k \neq X_k'] = \sum_{k=1}^{\infty} P[\|X_k\| > k]$$

$$= \sum_{k=1}^{\infty} E\{I[\|X_k\| > k]I[\|X_k\| \geq \alpha_k] + I[\|X_k\| > k]I[\|X_k\| < \alpha_k]\}$$

$$\leq \sum_{k=1}^{\infty} P[\|X_k\| \geq \alpha_k] + 2\sum_{k=1}^{\infty} k^{-2p}\alpha_k^p E\|X_k\|^p < \infty ,$$

we see that the sequences $\{X_n, n \geq 1\}$ and $\{X_n', n \geq 1\}$ are equivalent in the sense of Khinchin. Hence, we conclude that

$$n^{-1}\Big\|\sum_{j=1}^{n} X_j - EX_j'\Big\| \to 0 \text{ a.s. },\ n \to \infty .$$

Hence, for any given $\varepsilon > 0$

$$P\Big[n^{-1}\Big\|\sum_{j=1}^{n} X_j - EX_j'\Big\| \geq \varepsilon\Big] \to 0 ,\quad n \to \infty .$$

But, by (2.1) and (B), we get

$$P\left[n^{-1}\left\|\sum_{j=1}^{n} X_j\right\| \geq \varepsilon\right] \leq n^{-p}\varepsilon^{-p}E\left\|\sum_{j=1}^{n} X_j\right\|^p \leq An^{-p}\varepsilon^{-p}\sum_{j=1}^{n} E\|X_j\|^p \to 0,\ n \to \infty.$$

Therefore, we see that $n^{-1}\|\sum_{j=1}^{n} EX'_j\| \to 0$, $n \to \infty$, and consequently we have completed the proof of (2.2).

(iii)$\Rightarrow$(iv). Let $X_j = x_j\varepsilon_j$. Then under assumptionns (A_1)–(D_1) of (iv) we see that the conditions (A)–(D) of (iii) are satisfied. Hence, by (2.2),

$$n^{-1}\sum_{j=1}^{n}\varepsilon_j x_j \to 0 \text{ in probability },\ n \to \infty .$$

(iv)$\Rightarrow$(vi). Assume that (iv) holds and let $\{x_n, n \geq 1\}$ be a sequence of $\mathcal{X}$–valued elements such that $\sum_{j=1}^{\infty} j^{-p}\|x_j\|^p < \infty$. Then

$$\sum_{j=2}^{\infty} j^{-2p}\|x_j\|^p \sum_{i=1}^{j-1}\|x_i\|^p < \sum_{j=2}^{\infty} j^{-p}\|x_j\|^p < \infty ,$$

and $n^{-p}\sum_{i=1}^{n}\|x_i\|^p = o(1)$, so (A_1) and (B_1) are satisfied respectively.

Letting now $\alpha_n = n$, $n \geq 1$, we see that

$$\sum_{n=1}^{\infty} n^{-2p}\alpha_n^p\|x_n\|^p = \sum_{n=1}^{\infty} n^{-p}\|x_n\|^p < \infty ,$$

or (D_1) holds. Moreover, we conclude that $\|x_n\|/n \to 0$, $n \to \infty$, so only for finitely many n we have $\|x_n\| \geq n$, i.e. (C_1) is satisfied. Therefore we see that

$$n^{-1}\sum_{j=1}^{n}\varepsilon_j x_j \to 0 \text{ in probability },\ n \to \infty .$$

(ii)$\Rightarrow$(vii). Let $\{X_n, n \geq 1\}$ be any sequence of independent, symmetric $\mathcal{X}$–valued random elements such that (A_2)–(D_2) of (vii) are satisfied. Taking into account that $\{X_n, n \geq 1\}$ is a sequence of symmetrically distributed random variables we see that $\{X_n, n \geq 1\}$ and $\{\varepsilon_n X_n, n \geq 1\}$ are equidistributed. It follows that $\{X_n, n \geq 1\}$ satisfies SLLN iff $\{\varepsilon_n X_n, n \geq 1\}$ does. The Fubini's theorem shows that $\{\varepsilon_n X_n, n \geq 1\}$ satisfies SLLN iff $\{\varepsilon_n(\cdot)X_n(\omega), n \geq 1\}$ satisfies SLLN for almost all $\omega \in \Omega$.

Choose any ω satisfying the conditions (A_2)–(D_2). Then put $x_n = X_n(\omega)$, $n \geq 1$, and write

$$x'_n = \begin{cases} x_n & \text{if } \|x_n\| < n \\ 0 & \text{if } \|x_n\| \geq n \end{cases}$$

Since $\mathcal{X}$ is a Banach space of type p, we get using (1.1) that for any given $\varepsilon > 0$

$$(2.6)\qquad P\Big[n^{-1}\Big\|\sum_{i=1}^{n}\varepsilon_i x_i'\Big\| > \varepsilon\Big] \le \varepsilon^{-p}n^{-p}E\Big\|\sum_{i=1}^{n}\varepsilon_i x_i'\Big\|^p \le An^{-p}\sum_{i=1}^{n}\|x_i\|^p = o(1)\,.$$

Now put $S_n' = \sum_{j=1}^{n}\varepsilon_j x_j'$ and define

$$Y_{n,i} = E(\|S_n'\|\,\mathcal{F}_i) - E(\|S_n'\|\,|\mathcal{F}_{i-1})\,,$$

where $\mathcal{F}_0 = \{\emptyset,\Omega\}$, $\mathcal{F}_i = \sigma(\varepsilon_1 x_1',\ldots,\varepsilon_i x_i')$. We see that

$$\|S_n'\| - E\|S_n'\| = \sum_{i=1}^{n} Y_{n,i}\,.$$

Following the considerations given in the proof of the implication (ii)⇒(iii) and taking into account that now $E\|X_i\|^p = \|x_i\|^p$ we obtain the statement of (vii).

(vii)⇒(vi). Letting $X_i = \varepsilon_i x_i$, where $\{x_i, i \ge 1\}$ is a sequence of $\mathcal{X}$-valued elements such that $\sum_{j=1}^{\infty} j^{-p}\|x_j\|^p < \infty$, we see that the conditions (A_2)–(D_2) of (vii) are satisfied. Therefore, by(vii) $n^{-1}\sum_{j=1}^{n}\varepsilon_j x_j \to 0$ a.s., $n \to \infty$, which implies convergence in probability and ends the proof of Theorem 2.3.

A more general and stronger version of Theorem 2.3, is as follows.

Theorem 2.4. *Let* $1 \le p \le 2$. *The statements* (i), (ii), (iv), (v), (vi) *of Theorem* 2.2 *are equivalent to the following*

(iii') $n^{-1}\|\sum_{j=1}^{n} X_j - EX_jI[\|X_j\| < j]\,\| \to 0$ a.s., $n \to \infty$ *for all sequences* $\{X_n, n \ge 1\}$ *of independent random elements with* $EX_i = 0$, $i \ge 1$ *and for every nonnegative, even, continuous function* $\phi : \mathbf{R} \to \mathbf{R}^+$ *nondecreasing on* $(0,\infty)$, *with* $\lim_{n\to\infty}\phi(x) = \infty$ *and such that*

(a) $\phi(x)/x \searrow$ *as* $x \to \infty$

$$(A_3)\qquad \sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E\frac{\phi^p(\|X_{j_k}\|)}{\phi^p(j_k)+\phi^p(\|X_{j_k}\|)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^p\cdot E\frac{\phi^p(\|X_{j_{k-1}}\|)}{\phi^p(j_{k-1})+\phi^p(\|X_{j_{k-1}}\|)}\cdots\sum_{j_1=1}^{j_2-1} j_1^p\cdot E\frac{\phi^p(\|X_{j_1}\|)}{\phi^p(j_1)+\phi^p(\|X_{j_1}\|)} < \infty$$

$$(B_3)\qquad n^{-p}\sum_{i=1}^{n} i^p E\frac{\phi^p(\|X_i\|)}{\phi^p(i)+\phi^p(\|X_i\|)} = o(1)\,,$$

(C$_3$) $\sum_{n=1}^{\infty} P[\|X_n\| \geq \alpha_n] < \infty$,

for some sequence $\{\alpha_n, n \geq 1\}$ *of positive numbers such that*

(D$_3$) $\sum_{n=1}^{\infty} \phi^p(\alpha_n) E \frac{\phi^p \|X_n\|}{\phi^{2p}(n) + \phi^{2p}(\|X_n\|)} < \infty$,

or

(b) $\phi(x)/x \nearrow$, $\phi(x)/x^p \searrow$ *as* $x \to \infty$

$$(\mathrm{A}_4) \quad \sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E \frac{\phi(\|X_{j_k}\|)}{\phi(j_k) + \phi(\|X_{j_k}\|)} \sum_{j_{k-1}=k-1}^{j_k - 1} j_{k-1}^p \cdot E \frac{\phi(\|X_{j_{k-1}}\|)}{\phi(j_{k-1}) + \phi(\|X_{j_{k-1}}\|)} \cdots \sum_{j_1=1}^{j_2-1} j_1^p \cdot E \frac{\phi(\|X_{j_1}\|)}{\phi(j_1) + \phi(\|X_{j_1}\|)} < \infty ,$$

$$(\mathrm{B}_4) \quad n^{-p} \sum_{i=1}^{n} i^p E \frac{\phi(\|X_i\|)}{\phi(i) + \phi(\|X_i\|)} = o(1) ,$$

and (C$_3$) *is satisfied for some numerical sequence* $\{\alpha_n, n \geq 1\}$ *with*

$$(\mathrm{D}_4) \quad \sum_{n=1}^{\infty} \phi(\alpha_n) E \frac{\phi(\|X_n\|)}{\phi^2(n) + \phi^2(\|X_n\|)} < \infty .$$

Proof. Using the properties of the function ϕ and following the proof of Theorem 2.3 and the proof of Theorem 2 of [4] one can verify the validity of Theorem 2.4.

By Theorem 2.3 we can have a stronger result than Theorem 2.3 of [3].

Theorem 2.5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent* $\mathcal{X}$*-valued random variables with mean 0. If* $\mathcal{X}$ *is of type p,* $1 \leq p \leq 2$*, and* $\{X_n, n \geq 1\}$ *satisfies*

(2.7) *For any given* $\varepsilon > 0$ *there exists a sequence* $\{\alpha_n, n \geq 1\}$ *of positive numbers such that*

(a) $\sum_{j=2}^{\infty} j^{-2p} \alpha_j^p \sum_{i=1}^{j-1} \alpha_i^p < \infty$,

(b) $n^{-p} \sum_{i=1}^{n} \alpha_i^p = o(1)$,

(c) $\sum_{i=1}^{\infty} (\alpha_i / i)^{2p} < \infty$,

and $E(\|X_i\| I[\|X_i\| \geq \alpha_i]) \leq \varepsilon$, $i \geq 1$.

Then

$$\lim_{n \to \infty} E\Big\{ n^{-1} \Big\| \sum_{j=1}^{n} X_j \Big\| \Big\} = 0 .$$

Proof. Put

$$X'_j = X_j I[\|X_j\| < \alpha_j], \quad X^*_j = X'_j - EX'_j .$$

Then

$$\|X^*_j\| \le 2\alpha_j . \tag{2.8}$$

Moreover, we see that the conditions *(a)–(c)* and (2.8) imply that the assumptions (A)–(D) of Theorem 2.3 are satisfied for the sequence $\{X^*_n, n \ge 1\}$. Hence by Theorem 2.3 and its proof we find that

$$n^{-1}\Big\|\sum_{j=1}^{n} X^*_j\Big\| \to 0 \text{ a.s.}, \ n \to \infty ,$$

and

$$n^{-1}E\Big\|\sum_{j=1}^{n} X^*_j\Big\| \to 0, \ n \to \infty .$$

Since $EX_j = 0$ we find

$$EX'_j = EX_j I[\|X_j\| \ge \alpha_j]$$

and so

$$\|EX^*_j\| = E\|X_j\| I[\|X_j\| \ge \alpha_j] < \varepsilon .$$

Moreover, we have

$$E\|X_j - X'_j\| = E\|X_j\| I[\|X_j\| \ge \alpha_j] < \varepsilon$$

and so

$$E\Big\|\sum_{j=1}^{n} X_j\Big\| \le E\Big\|\sum_{j=1}^{n} X'_j\Big\| + E\Big\|\sum_{j=1}^{n} EX^*_j\Big\| + E\Big\|\sum_{j=1}^{n}(X_j - X'_j)\Big\|$$

$$\le E\Big\|\sum_{j=1}^{n} X'_j\Big\| + \sum_{j=1}^{n}\|EX^*_j\| + \sum_{j=1}^{n} E\|X_j - X'_j\| .$$

Hence we get

$$\lim_{n\to\infty}\sup\Big\{n^{-1}E\Big\{\Big\|\sum_{j=1}^{n} X_j\Big\|\Big\} \le 3\varepsilon$$

and since $\varepsilon > 0$ is arbitrary we have proved

$$\lim_{n\to\infty} E\Big\{n^{-1}\Big\|\sum_{j=1}^{n} X_j\Big\|\Big\} = 0 .$$

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STRESZCZENIE

Niech $\{X_n, n \geq 1\}$ będzie ciągiem niezależnych elementów losowych przyjmujących wartości z przestrzeni Banacha $\mathcal{X}$. W pracy udowodniono, że teicherowska wersja mocnego prawa wielkich liczb zachodzi wtedy i tylko wtedy gdy $\mathcal{X}$ jest przestrzenią typu p. Powyższy rezultat rozszerza wyniki podane w [3] gdzie rozważana jest wersja Chunga mocnego prawa wielkich liczb.

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