ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA

VOL. XLV, 5 SECTIO A 1991

Department of Mathematics
University of South Florida

A. W. GOODMAN

The Valence of Certain Sums

Listność pewnych sum

Abstract. Let $\mathcal{A}$ be the collection of functions $f(z)$ regular in $E: |z| < 1$ and normalised by $f(0) = 0$ and $f'(0) = 1$, and set $F(z) = (f(z) + g(z))/2$. We investigate relations between k, m, and n, the valences in E of $(z), g(z)$ and $F(z)$, respectively.

1. Introduction. We consider a problem which we will denote by $V(k, m, n)$ where k, m, n are positive integers which may include ∞ as a positive integer.

Let

$$F(z) = \frac{1}{2}(f(z) + g(z)) , \tag{1}$$

where $f(z)$ and $g(z)$ are functions in $\mathcal{A}$ the set of normalized functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{2}$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n , \tag{3}$$

which are regular in $E : |z| < 1$.

Given k, m, and n, positive integers does there exist functions f, g and F such that f has valence k in E, g has valence m in E, and F has valence n i E. We denote this problem by the symbol $V(k, m, n)$ which is a function with the range {Yes, No}.

What may appear to be the hardest case, $V(1, 1, infty)$ was solved affirmatively in [1]. In this paper, we look at various other combinations of k, m, and n. In the last section, we suggest various extensions and generalizations of the problem $V(k, m, n)$.

2. Some special cases. First we note that $V(k, m, n) = V(m, k, n)$ so that W.L.O.G. we can always assume that $k \le m$.

Theorem 1. *The problem $V(k, k, 1)$ has a solution for each $k = 1, 2, \ldots, \infty$.*

Proof. Let

$$f(z) = z + h(z)\,, \quad g(z) = z - h(z) \tag{4}$$

where

$$h(z) = \sin(z^2/(1-z^2)) = \frac{z^2}{1-z^2} - \frac{1}{6}\,\frac{z^6}{(1-z^2)^3} + \dots \tag{5}$$

so that f and g have the requires normalization. Further, $F(z) = (f(z)+g(z))/2 \equiv z$ which is trivially univalent in E. Next, the function $u(z) = z^2/(1-z^2)$ maps the interval $(0,1)$ onto $(0,\infty)$ in a 1–1 continuous manner. Hence there is a sequence $0 < z_1 < z_2 < \ldots < z_q < \ldots < 1$ such that $u(z_{2q+1}){=}2q\pi+\pi/2$ and $u(z_{2q}){=}2q\pi-\pi/2$ for $q = 1, 2, \ldots$ Then $\sin u(z)$ alternates between $+1$ and -1 and hence has infinitely many zeros. The function $z + \sin u(z)$ is positive at z_{2q+1} and negative at z_{2q} so $f(z)$ is ∞–valent in E. The same type of argument shows that $g(z) = z - h(z)$ is also ∞–valent in E. Thus the problem $V(\infty,\infty,1)$ has a solution.

For finite k, one can consider $f(rz)/r$ and $g(rz)/r$ for suitable r, but it is simpler to set $f(z) = z + Az^k$ and $g(z) = z - Az^k$ with sufficiently large A. Then $V(k,k,1)$ has a solution for each finite k, $k \geq 1$.

Theorem 2. *The problem $V(k,k,,n)$ has a solution for each $n = 2,3,\ldots,\infty$, and each $k = 2,3,\ldots,\infty$.*

This theorem extends Theorem 1 from the case $n = 1$ to the case $n > 1$.

Proof. First $V(\infty,\infty,\infty)$ is trivial. Set $f(z) = g(z) =$ any normalized infinite–valent function.

Next consider $V(\infty,\infty,n)$ with n finie. In this case, set

$$f(z) = z + Az^n + B\sin u(z) \tag{6}$$

$$g(z) = z + Az^n - B\sin u(z) \tag{7}$$

where (as in Theorem 1) $u(z) = z^2/(1-z^2)$. If $A > 1$, then $(f(z)+g(z))/2$ is n–valent. Further, if $B > 1 + A$, then the argument used in Theorem 1 shows that $f(z)$ and $g(z)$ are both infinite–valent. Hence the problem $V(\infty,\infty,n)$ has a solution for each finite n.

For $V(k,k,n)$ with $1 < k < n < \infty$, set

$$f(z) = z + Az^k + Bz^n \tag{8}$$

$$g(z) = z - Az^k + Bz^n \tag{9}$$

with $B > 1$ and $A > 1 + B$.

To complete the proof of Theorem 2, we need to settle the cases $V(k,k,\infty)$ with $k > 1$ and $V(k,k,n)$ with $n < k$.

For $V(k,k,\infty)$ set

$$f(z) = \frac{e^{i\alpha}}{2A}\left[-1 + \exp\left(Ae^{i\alpha}\ln\frac{1+z}{1-z}\right)\right]\,, \tag{10}$$

where $A > 0$ and $0 < \alpha < \pi/2$. Further set

$$g(z) = \frac{e^{i\beta}}{2B}\left[-1 + \exp\left(Be^{-i\beta}\ln\frac{1+z}{1-z}\right)\right], \tag{11}$$

where $B > 0$ and $0 < \beta < \pi/2$. Since $Ae^{i\alpha}\ln((1+z)/(1-z))$ maps E onto an infinite strip of width $A\pi$ that makes an angle α with the positive real axis, then $f(z)$ is k–valent if $2(k-1) < A/\cos\alpha < 2k$. A similar remark holds for $g(z)$ which is m–valent if $2(m-1) < B/\cos\beta < 2m$.

Now set $k = m$, $A = B$, and $\alpha = \beta$ with $(2k-1)\cos\alpha = A$. Then both $f(z)$ and $g(z)$ are k–valent in E. Further, $F(z) = (f(z) + g(z))/2$ gives

$$F(z) = C_0 + \frac{1}{4A}Q(z) \tag{12}$$

where $C_0 = -(\cos\alpha)/2A$ and

$$Q(z) = e^{-i\alpha}\exp\left(Ae^{i\alpha}\ln\frac{1+z}{1-z}\right) + e^{i\alpha}\exp\left(Ae^{-i\alpha}\ln\frac{1+z}{1-z}\right). \tag{13}$$

A small computation will show that $Q(z) = 0$ whenever

$$\ln\frac{1+z}{1-z} = \frac{(2q+1)\pi + 2\alpha}{2A\sin\alpha}, \tag{14}$$

where q is any integer. Since this equation is satisfied for some z in E for each integer q, it follows that $F(z)$ is ∞–valent in E.

For $V(k,k,n)$ with $1 < n < k < \infty$ set

$$f(z) = z + Az^n + Bz^k, \tag{15}$$

and

$$g(z) = z + Az^n - Bz^k, \tag{16}$$

with $A > 1$ and $B > 1 + A$.

This analysis omits the case $V(1,1,n)$ where n is finite. It is clear that equations (8) and (9) cannot be used when $k = 1$. The transformation $T(z) = \varphi(rz)/r$ applied to f, g and F in equations (10) and (11) will yield and F with exactly n zeros when r is selected properly. However, this does not mean that $F(rz)/r$ is n–valent, and it is possible that as r decreases from 1 to 0, the valence of $F(rz)/z$ may have jump discontinuities greater than 1. For the moment, the problem $V(1,1,n)$ remains unsettled.

3. The unsymmetrical cases. Here we consider $V(k,m,n)$ where $k < m$, and k, m, n are any positive integers including ∞.

Theorem 3. *The problem $V(k,m,\infty)$ has a solution for each pair of positive integers k and m.*

Proof. The case $k = m$ was settled in Theorem 2. By the symmetry of the problem, we may assume W.L.O.G. that $1 \le k < m$. We use the two functions $f(z)$ and $g(z)$ defined by equations (10) and (11) where we may have $A \neq B$ and $\alpha \neq \beta$. Then $F(z) = ((f(z) + g(z))/2$ gives $F(z) = C_0 + Q(z)$ where now

$$C_0 = -\frac{e^{-i\alpha}}{4A} - \frac{e^{i\beta}}{4B}, \tag{17}$$

and

$$Q(z) = \frac{e^{-i\alpha}}{4A}\exp\Big(Ae^{i\alpha}\ln\frac{1+z}{1-z}\Big) + \frac{e^{i\beta}}{4B}\exp\Big(Be^{-i\beta}\ln\frac{1+z}{1-z}\Big). \tag{18}$$

A small computation shows that $Q(z) = 0$ whenever

$$\ln\frac{1+z}{1-z} = \frac{((2q+1)\pi + \alpha + \beta)i + \ln(A/B)}{Ae^{i\alpha} - Be^{-i\beta}}. \tag{19}$$

Now $f(z)$ and $g(z)$ will be k-valent and m-valent respectively if A, α, B and β satisfy the conditions

$$A = (2k-1)\cos\alpha, \quad \text{and} \quad B = (2m-1)\cos\beta\,. \tag{20}$$

To simplify (19), we impose the further condition

$$A\cos\alpha = B\cos\beta\,. \tag{21}$$

Then $Q(z) = 0$ whenever

$$\ln\frac{1+z}{1-z} = \frac{(2q+1)\pi + \alpha + \beta}{A\sin\alpha + B\sin\beta} - \frac{\ln(A/B)}{A\sin\alpha + B\sin\beta}\,i\,. \tag{22}$$

The right side of (22) gives an infinite set of points that lie on a parallel to the real axis and hence for each integer q, there is a z in E that satisfies (22) if

$$\left|\frac{\ln(A/B)}{A\sin\alpha + B\sin\beta}\right| < \frac{\pi}{2}\,. \tag{23}$$

Since $m > k \ge 1$, the conditions (20) and (21) will give

$$1 < \frac{2m-1}{2k-1} = \frac{\cos^2\alpha}{\cos^2\beta}, \tag{24}$$

and hence $1 < \cos\alpha/\cos\beta = B/A$, so $B > A$. Now $A\sin\alpha > 0$ so (23) is satisfied if

$$\frac{\ln(B/A)}{B\sin\beta} < \frac{\pi}{2}\,. \tag{25}$$

We select $\alpha = 0$. Then (20) dictates that $A = 2k-1$ and (21) gives $\cos\beta = A/B$. From $\sin^2\beta = 1 - (A/B)^2$ and a little manipulation we see that (25) is satisfied if

$$\frac{\pi}{2} > \frac{1}{A}\frac{\ln(B/A)}{\sqrt{(B/A)^2 - 1}} \equiv \frac{1}{A}\frac{\ln t}{\sqrt{t^2-1}} \equiv \frac{1}{A}I_*(t) \tag{26}$$

where $t = B/A > 1$. A computer program shows that for $1 < t < \infty$ we have $\max I_*(t) \approx 0.4037 < \pi/2$ when $t \approx 2.2185$, the only zero of $I_*'(t)$. Thus $f(z)$ has infinitely many zeros in E.

The other unsymmetrical cases $V(k, \infty, n)$ and (k, m, n) where k, m, n are all finite, seem to be more difficult.

4. Generalizations. A number of generalizations may be of interest. First, suppose that we fix positive numbers α, β with $\alpha + \beta = 1$ and replace (1) by

$$F(z) = \alpha f(z) + \beta g(z) . \tag{27}$$

Then $V(k, m, n)$ denotes the problem: are there functions in A for which f, g, and F have calence k, m, and n respectively.

An affirmative answer in the simplest case $V(1, 1, \infty)$ was obtained in [2], but in that work, α and β were restricted to the interval $\left(1/(1+e^\pi), e^\pi/(1+e^\pi)\right)$. Later, Ji [3] extended this result to the full interval $(0, 1)$.

The same questions $V(k, m, n)$ can be asked if

$$F(z) = f^\alpha(z) g^\beta(z) \equiv z \left[\frac{f(z)}{z}\right]^\alpha \left[\frac{g(z)}{z}\right]^\beta . \tag{28}$$

Here, of course, we must restrict f, g and F to be the functions with only one zero in E the one of the origin. Again, the simplest case $V(1, 1, \infty)$ was settled in [2] in the affirmative for α and β in a certain interval and again Ji [3] extended the result to all α, β in $(0, 1)$.

Other compositions, such as

$$F(z) = \sum_1^\infty a_n b_n z^n , \quad \text{or} \quad F(z) = \sum_1^\infty \frac{a_n b_n}{n} z^n , \tag{29}$$

may also yield interesting problems.

The methods used in Sections 2 and 3 may settle some of the easier cases, but a complete analysis of all cases $V(k, m, n)$ in any one of the above problems may be difficult.

REFERENCES

[1] Goodman, A. W., *The valence of sums and products*, Canadian Jour. of Math. 20 (1968), 1173–1177.

[2] Goodman, A. W., *The valence of certain means*, Jour d'Analuse Math. 22 (1969), 355–361.

[3] Ji, Zhou, *A note on the valence of certain means*, (Submitted) Journal Sichuan Normal University (in Chinese) and Proc. Amer. Math. Soc.

STRESZCZENIE

Niech $\mathcal{A}$ oznacza rodzinę funkci $f(z)$ regularnych w $E = \{z : |z| < 1\}$ i unormowanych; $f(0) = 0$, $f'(0) = 1$. Oznaczmy $F(z) = \frac{f(z)+g(z)}{2}$, $f, g \in \mathcal{A}$ W pracy badane są zależności pomiędzy listnościami w E funkcji $f(z)$, $g(z)$ i $F(z)$.

(received February 11, 1992)