ANNALES UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. XLV, 3 SECTIO A 1991

Department of Mathematics, University of Mississippi
Instytut Matematyki, UMCS
Department of Mathematics, University of Cincinnati

W. M. CAUSEY , J. G. KRZYŻ , E. P. MERKES

Some Remarks on a Distortion Lemma

Kilka uwag dotyczących pewnego lematu o zniekształceniu

Abstract. The authors consider for $|z| = r$ bounds on $|z/f(z) - 1|$ over the class S of all normalized analytic univalent functions f. In particular, they show that the r.h.s. in (1) should be replaced by $2r + 3r^2$. The estimates of $|z/f(z) - 1|$ play a role in the determination of the choice of α for the univalence of the integral transform $\int_0^z [f(t)/t]^\alpha dt$ when $f \in S$. Since (1) is not valid for all $z \in \mathbf{D}$, the known bound on $|\alpha|$ remains at 1/4.

1. Introduction. Let S denote the class of normalized analytic univalent functions in the open unit disk $\mathbf{D}$ and let α be a fixed complex number. For many years two of the present authors, as well as many others, have attempted to find the choices of complex α such that the function $G(z) = \int_0^z [g(t)/t]^\alpha dt$ is in S whenever g is in S (cf. e.g. [1], [2], [4], [5]). The best known result is $|\alpha| \le \frac{1}{4}$ which was first published in 1972 [4]. A result of Royster [9] proves that the modulus of α cannot exceed 1/2 and, in fact, G is in S for all α, $|\alpha| \le \frac{1}{2}$, provided g is in addition starlike, cf. [5].

In a recent article [6], J. Miazga and A. Wesołowski attempt to prove the bound on $|\alpha|$ is 1/3. Their proof is based on what appears to be a nice general result.

Lemma A [6]. *If f is in S, then for fixed z in $\mathbf{D}$ the inequality*

$$|z/f(z) - 1| \le 2r + r^2\,, \quad |z| = r\,, \tag{1}$$

holds. The Koebe function $f(z) = z(1+z)^{-2}$ establishes sharpness.

This lemma, however, is incorrect and, as a consequence, the known bound on $|\alpha|$ remains at 1/4. Using a classical 1932 result of Grunsky , cf. e.g. [3, p.323], which is quoted here as Lemma B, it is easily verified that

$$\sup\{|z/f(z) - 1| : f \in S,\ z \in \mathbf{D}\} = 5 \tag{2}$$

and this implies Lemma A, as stated, is incorrect.

Nonetheless, the inequality (1) is indeed true if we restrict f to be in the subclass S^* of starlike functions in S (Proposition 2). It is also true for $f \in S$ and r sufficiently small. However, the inequality (1) must be replaced for arbitrary $f \in S$, $z \in \mathbf{D}$, by

$$(3) \qquad |z/f(z) - 1| \leq 2r + 3r^2, \quad |z| = r .$$

2. Bounds on $|z/f(z) - 1|$. We first quote the classical result of Grunsky as

Lemma B. *For each z, $|z| = r < 1$, the region $\{\log f(z)/z : f \in S\}$ is the disk*

$$(4) \qquad \left\{\zeta : |\zeta + \log(1 - r^2)| \leq \log \frac{1+r}{1-r}\right\} .$$

As an immediate consequence of this result we obtain

Proposition 1. *The region $\{z/f(z) : f \in S, \ z \in \mathbf{D}\}$ is the punctured disk $\{w : 0 < |w| < 4\}$.*

In fact, by (4) with $z \in \mathbf{D}$ and the natural branch of the complex logarithm we have $\{\log z/f(z) : f \in S \text{ and } |z| < 1\} = \{\zeta : \operatorname{Re} \zeta < \log 4\}$ and the Proposition follows by exponentiation.

If we take $w = -4 + \varepsilon$, where $0 < \varepsilon < 1$, then we can find $f \in S$ and $z \in \mathbf{D}$ so that $z/f(z) = w$. Hence $|w - 1| = 5 - \varepsilon$ and we conclude that $|z/f(z) - 1| < 5$ for z in $\mathbf{D}$ and 5 is the best possible bound. This shows that the inequality (1) is incorrect.

Nonetheless, if we restrict f to be in the subclass S^* of starlike functions in S, Lemma A is indeed true.

Proposition 2. *If f is in S^*, then for a fixed z in $\mathbf{D}$ the inequality (1) holds. Equality holds in (1) if and only if $f(z) = z(1 + e^{i\theta}z)^{-2}$, θ real, i.e. a Koebe function.*

Proof. It is a well-known result due to A. Marx and E. Strohhäcker (cf. e.g. [8, p.50]), that for a fixed z, $|z| = r < 1$, and $f \in S^*$ the point $w = [z/f(z)]^{1/2}$ ranges over the disk $|w - 1| \leq r$. Furthermore equality holds if and only if $f(z) = z(1 + e^{i\theta}z)^{-2}$, θ real. Thus $[z/f(z)]^{1/2} = 1 + \rho e^{i\theta}$, where $|z| \leq r$ and $\rho \leq r$. This implies that $z/f(z) - 1 = 2\rho e^{i\theta} + \rho^2 e^{2i\theta}$ and the Proposition follows.

As observed by P. Pawłowski in a paper to be published in this volume, the inequality (1) is also true for close-to-convex functions.

From Lemma B we can obtain for $|z| \leq r < 1$ a sharp inequality for the supremum of the expression on the left in (1) for all $f \in S$. Unfortunately the result is rather complicated and implicit. Indeed, the boundary of the range of $z/f(z)$, for $f \in S$, $|z| = r$, can by (4) be parametrized as

$$w = w_r(t) = A(t)\big(\cos\psi(t) + i \sin\psi(t)\big) , \quad -\pi \leq t \leq \pi ,$$

where

$$(5) \qquad A(t) = A_r(t) = (1 - r^2)\Big(\frac{1+r}{1-r}\Big)^{\cos t} , \quad \psi(t) = \psi_r(t) = \sin t \log \frac{1+r}{1-r} .$$

By a standard calculus argument, we obtain the following

Theorem 1. *If f is in S and $|z| = r < 1$, then*

$$\left|\frac{z}{f(z)} - 1\right| \le \left[A^2(t_0) - 2A(t_0)\cos\psi(t_0) + 1\right]^{1/2}, \tag{6}$$

where A, ψ are defined by (5) and $t_0 = t_0(r)$ is a suitable zero of the function

$$D_r(t) = \sin(t + \psi_r(t)) - A_r(t)\sin t \tag{7}$$

For each $r \in (0;1)$ there is a function in S such that the equality holds in (6).

Due to symmetry we may assume that $0 \le t \le \pi$. Obviously the end points of the interval $(0;\pi)$ are zeros of (7). However, $\cos\psi(t) = 1$ for $t = 0, \pi$ and so the r.h.s. in (6) becomes $|A(t) - 1|$. Since $|A(\pi) - 1| = 2r - r^2 < |A(0) - 1| = 2r + r^2$, the case $t_0 = \pi$ must be rejected.

Numerical work using MAPLE indicates that the only zeros of $D_r(t)$ on the interval $[0;\pi]$ are the end points (and so $t_0 = 0$) when $r \le 0.819497$. When $|z| = r$ is restricted to this range, the Koebe function is extremal and (1) is correct. For $r \ge 0.819498$, however, $D_r(t)$ has a finite number of additional zeros and, in particular, $0 < t_0 < \pi$. When $r = 0.95$, for example, t_0 is approximately equal to 0.32142 and the bound on the right in (6) is approximately 2.8987.

Although Theorem 1 gives sharp bounds, it depends on the deep theorem of Grunsky quoted as Lemma B and the final result is implicit. There is a simpler, explicit, and more attractive, although less sharp, form that can be proved by elementary methods. At the same time it is a correct version of Lemma A with the majorant being a polynomial in r of degree at most 2. We have the following

Proposition 3. *If $f(z) = z + a_2 z^2 + \dots$ is in S and $0 < |z| = r < 1$, then*

$$\left|\frac{z}{f(z)} - 1\right| < |a_2| r + 3r^2 \le 2r + 3r^2 . \tag{8}$$

Proof. If $f \in S$, then $h(\zeta) \equiv 1/f(z)$, $\zeta = 1/z$, is in the familiar class Σ of meromorphic univalent functions and $h(\zeta) \ne 0$ for $|\zeta| > 1$. We have

$$h(\zeta) = \zeta + b_0 + b_1/\zeta + \dots = b_0 + h_0(\zeta) .$$

Now, we have $|b_0| \le 2$ for a non-vanishing $h \in \Sigma$ and $|h_0(\zeta) - \zeta| < 3|\zeta|^{-1}$ for $h_0(\zeta) = \zeta + b_1/\zeta + \dots$, cf. [7, p.25 (Ex.139, 144)]. We conclude

$$|h(\zeta) - \zeta| = |h_0(\zeta) - \zeta + b_0| < |b_0| + 3/|\zeta|$$

and since $b_0 = -a_2$, $1/|\zeta| = r$, we have

$$\left|\frac{z}{f(z)} - 1\right| = \left|\frac{h(\zeta)}{\zeta} - 1\right| < \frac{|b_0|}{|\zeta|} + \frac{3}{|\zeta|^2} = |a_2| r + 3r^2 \le 2r + 3r^2 .$$

The bound (8) is sharp in the limit as $r \to 1$.

When $r = 0.95$, we obtain the value 4.6075 for this bound while the sharp bound in Theorem 1 is less than 2.8988.

Note that $|a_2|r + 3r^2 = 2r + r^2 + 2r[r - (1 - \frac{1}{2}|a_2|)] < 2r + r^2$ for $0 < r < 1 - \frac{1}{2}|a_2|$. This establishes

Corollary 1. If f is not a Koebe function, then (1) holds for all $|z| = r$ in the interval $(0; 1 - \frac{1}{2}|a_2|)$.

In particular, (1) is valid for $z \in \mathbf{D}$ if $f \in S$ and $f''(0) = 0$. By the argument in [6], we have a new result on the integral transform:

Corollary 2. Let $g(z) = z + a_3 z^3 + a_4 z^4 + \dots$ be in S. Then $G(z) = \int_0^z [g(t)/t]^\alpha dt$ is also in S if $|\alpha| \le 1/3$.

3. Concluding remarks. In [6] the authors by variational methods essentially prove the cited result of Grunsky but state that the expression (6) is maximized when $t_0 = 0$. The latter is not always the case. The remaining arguments in their paper are all valid but, for the full class S, 1/5 is the best bound for $|\alpha|$ we can obtain by their argument and the corrected Lemma A, i.e. the inequality (8).

REFERENCES

[1] Causey, W. M., *The close-to-convexity univalence and integral*, Math. Z., 99 (1967), 207–212.

[2] Causey, W. M., *The univalence of an integral*, Proc. Amer. Math. Soc., 3 (1971), 500–502.

[3] Duren, P. L., *Univalent Functions*, Springer Verlag, 1983.

[4] Kim, Y. J., Merkes, E. P., *On the integral of powers of a spirallike function*, Kyungpook Math. J., 12 (1972), 249–253.

[5] Merkes, E. P., *Univalence of an integral transform*, Contemporary Math., Amer. Math. Soc., 38 (1985), 113–119.

[6] Miazga, J., Wesołowski, A., *On univalence of a certain integral*, Zeszyty Nauk. Politech. Rzeszowskiej, Mat. Fiz. 9 (1989), 25–31.

[7] Pólya, G., Szegö, G., *Aufgaben und Lehrsätze aus der Analysis*, Vol.II, Dover Publ., New York 1945.

[8] Pommerenke, Chr., *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen 1975.

[9] Royster, W. C., *On the univalence of a certain integral*, Mich. Math. J. 12 (1965), 385–387.

STRESZCZENIE

W pracy tej rozważane są oszacowania wyrażenia $|z/f(z) - 1|$ dla $|z| = r$ w klasie S unormowanych funkcji jednolistnych f. W szczególności wykazano, że w nierówności (1) należy zastąpić prawą stronę przez wyrażenie $2r + 3r^2$. Oszacowania wyrażenia $|z/f(z) - 1|$ są wykorzystywane przy wyznaczaniu liczb α takich, że transformacja całkowa $S \ni f \mapsto \int_0^z [f(t)/t]^\alpha dt$ zachowuje jednolistność. Ponieważ nierówność (1) nie jest spełniona dla wszystkich $z \in \mathbb{D}$, więc znane oszacowanie na $|\alpha|$ równe $1/4$ nadal pozostaje w mocy.

(received September 23, 1991)