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On a Problem of M. Biernacki for the Class of Close-to-star Functions

O pewnym problemie M. Biernackiego dla klasy funkcji prawie gwiaździstych

Abstract. In this paper we investigate the relationship between the subordination and the majorization in the class H^* of close-to-star functions.

1. Introduction. Let $\mathbf{C}$ denote the complex plane, $K_r = \{z \in \mathbf{C} : |z| < R\}$, $K_1 = K$ and f, F be two holomorphic functions in the disk K_R such that $f(0) = F(0)$.

Definition 1. The function f is subordinate to F in the disk K_R, if there exists a function ω holomorphic in K_R such that $|\omega(z)| \le |z|$ and $f(z) = F(\omega(z))$ for $z \in K_R$. In this case we write $f \prec F$ in K_R.

Definition 2. The function f is majorized by F in the disk K_R if $|f(z)| \le |F(z)|$ for every $z \in K_R$. Then we write $f \ll F$ in K_r.

It means that there exists in K_R a holomorphic function Φ such that $|\Phi(z)| \le 1$ for $z \in K_R$ and $f(z) = \Phi(z)F(z)$ for every $z \in K_R$.

The investigations concerning relations between the subordination in the unit disk K and the majorization in some smaller disk were initiated by M. Biernacki. The problem posed by Biernacki may be formulated in the following way. Determine the greatest possible number $r_0 = r_0(\mathcal{H}, \mathcal{F}) \in (0,1)$ such that for every pair of functions $f \in \mathcal{H}$, $F \in \mathcal{F}$ (where $\mathcal{H}, \mathcal{F}$ are fixed classes of holomorphic functions in the disk K) the implication

$$f \prec F \text{ in } K \Longrightarrow f \ll F \quad \text{in the disk } K_{r_0}$$

holds. The number r_0 is called the radius of majorization in Biernacki problem for the pair of classes $\mathcal{H}, \mathcal{F}$. This problem was investigated by many authors for various classes of holomorphic functions usually under the assumption of univalence. An interesting result for the class of typically real functions was obtained by W. Szapiel [4]. The function F holomorphic in K and such that $F(0) = 0$, $F'(0) = 1$, is said to be typically real if it takes real values on the segment $(-1,1)$ of the real axis and satisfies the condition $\operatorname{Im} z \cdot \operatorname{Im} F(z) > 0$ for $z \in K \setminus (-1,1)$. The class of typically real functions will be denoted by TR. Szapiel has determined the radius of majorization in the class TR under the assumption that the minorant f is typically real. In this case $r_0' = 0.3637\ldots$.

2. The majorization in the class of close–to–star functions. We denote by S^* the well-known class of starlike functions with the usual normalization. The function F holomorphic in the disk K and such that $F(0) = 0$, $F'(0) = 1$, is said to be close–to–star if there exists a function $G \in S^*$ such that

$$\operatorname{Re} \frac{F(z)}{G(z)} > 0 \quad \text{for } z \in K . \tag{1}$$

The class of close–to–star functions is denoted by H^*.

It is easy to observe that if $G(z) = \dfrac{z}{1-z^2}$ and the coefficients of the function F are real, then the condition (1) takes the form

$$\operatorname{Re}\left\{(1-z^2)\frac{F(z)}{z}\right\} > 0 .$$

This means that $F \in TR$ and hence $TR \subset H^*$.

The condition (1) can be written in an equivalent form

$$F(z) = G(z)p(z) , \quad z \in K , \tag{2}$$

where $p \in \mathcal{P}$, $\mathcal{P}$ being the class of functions p holomorphic in K and such that $\operatorname{Re} p(z) > 0$ for $z \in K$ and $p(0) = 1$.

Further on we will use the following four results quoted as Lemmas.

Lemma A [2]. *If $G \in S^*$ and x, z are arbitrary numbers from the disk K, then we have the estimate*

$$\left|\frac{G(x)}{G(z)}\right| \le \frac{\dfrac{|x|}{1-|x|^2}}{\dfrac{|z|}{1-|z|^2}} \cdot \frac{1 + \left|\dfrac{x-z}{1-\bar{z}x}\right|}{1 - \left|\dfrac{x-z}{1-\bar{z}x}\right|} . \tag{3}$$

Lemma B [1]. *Let $p \in \mathcal{P}$. Then for any x, z in the disk K the following estimate*

$$\left|\frac{p(x)}{p(z)}\right| \le \frac{1 + \left|\dfrac{x-z}{1-\bar{z}x}\right|}{1 - \left|\dfrac{x-z}{1-\bar{z}x}\right|} \tag{4}$$

holds.

Lemma C. *If $F \in H^*$ and x, z are arbitrary numbers in the disk K, then we have*

$$\left|\frac{F(x)}{F(z)}\right| \le \frac{\dfrac{|x|}{1-|x|^2}}{\dfrac{|z|}{1-|z|^2}} \cdot \left(\frac{1 + \left|\dfrac{x-z}{1-\bar{z}x}\right|}{1 - \left|\dfrac{x-z}{1-\bar{z}x}\right|}\right)^2 . \tag{5}$$

The above estimate (5) follows from the equality (2) and from the estimate (3), (4).

Lemma D [3]. *If f, F are holomorphic functions in K, $f(0) = F(0) = 0$, $\arg f'(0) = \arg F'(0)$ or $f'(0) = 0$ and $f \prec F$ in the disk K, then for a fixed z_0, $|z_0| = r_0 < 1$,*

$$|f(z_0)| \le \max\left\{\max_{|z|=r_0^2} |F(z)|, \max_{\gamma} |F(\gamma z_0)|\right\} \tag{6}$$

where γ ranges over all numbers of the form $\gamma = \dfrac{\alpha \pm i r_0}{1 \pm i\alpha r_0}$, $0 \le \alpha \le 1$.

We shall now prove the main result of this paper

Theorem . *Let $\mathcal{H}$ denotes the class of functions $f(z) = a_1 z + a_2 z^2 + \ldots$, $a_1 \ge 0$, holomorphic in K. If $f \in \mathcal{H}$, $F \in H^*$ and $f \prec F$ in K then $|f(z)| \le |F(z)|$ for $|z| \le \sqrt{5} - 2 \approx 0.236$. The equality takes place only for $f(z) \equiv F(z)$.*

Proof. Let z_0 be a fixed point from the disk K, $|z_0| = r$. It follows from Lemma D that $|f(z_0|$ cannot exceed the greater of two maxima in (6).

Consider the case the second maximum i.e. $\max\limits_{\gamma} |F(\gamma z_0)|$ is greater. Setting in the estimation (5) (Lemma C) $z = z_0$ and $x = z_0 \dfrac{\alpha + i\beta}{1 + i\beta\alpha}$, $0 \le \alpha \le 1$, $\beta = \pm r$, we obtain

$$\left|\frac{F(x)}{F(z_0)}\right| \le \frac{\sqrt{(\alpha^2 + r^2)(1 + \alpha^2 r^2)}}{(1 + r^2)} \cdot \frac{\left[\sqrt{(1 - r^2)^2 + r^2(1 - \alpha)^2} + (1 - \alpha)r\right]^4}{(1 - r^2)^4} .$$

Putting $a = r^2 + \frac{1}{r^2}$ we obtain

$$\left|\frac{F(x)}{F(z_0)}\right| \le \psi(a, \alpha) ,$$

where

$$\psi(a, \alpha) = \frac{\sqrt{\alpha^4 + a\alpha^2 + 1}\left[\sqrt{a - 2 + (1 - \alpha)^2} + (1 - \alpha)\right]^4}{\sqrt{a + 2}\,(a - 2)^2} , \quad a > 2 . \tag{7}$$

Let us note that $\psi(a, 1) = 1$ for every $a > 2$.

The function (7) as a function of α has the derivative at $\alpha = 1$ equal to

$$\psi'(a, 1) = 1 - \frac{4}{\sqrt{a - 2}} ,$$

which is negative for $a < 18$, i.e. $r > \sqrt{5} - 2$. This means that for $\alpha < 1$ sufficiently near 1 and for $a < 18$ we have $\psi(a, \alpha) > 1$. We show now that for $a \ge 18$ i.e.

$r \leq \sqrt{5} - 2$ and for every $\alpha \in \langle 0, 1 \rangle$ we have $\psi(a, \alpha) \leq 1$. The function (7) can be written in the form

(8) $$\psi(a,\alpha) = \sqrt{\frac{1 + a\alpha^2 + \alpha^4}{2 + a}} \left[\sqrt{1 + \frac{(1-\alpha)^2}{a-2}} + \frac{1-\alpha}{\sqrt{a-2}} \right]^4$$

for each $\alpha \in \langle 0, 1 \rangle$ and is decreasing as a function of a. Hence it follows that

$$\psi(a, \alpha) \leq \psi(18, \alpha)$$

for every $0 \leq \alpha \leq 1$. It suffices to show that

(9) $$\psi(18, \alpha) = \frac{\sqrt{1 + 18\alpha^2 + \alpha^4} \left[\sqrt{16 + (1-\alpha)^2} + 1 - \alpha \right]^4}{2\sqrt{5} \cdot 256} \leq 1$$

for every $\alpha \in \langle 0, 1 \rangle$. It should be noted that

$$\psi(18, 1) = 1 \,, \quad \psi(18, 0) = \frac{(\sqrt{17} + 1)^4}{2\sqrt{5} \cdot 256} < 1 \,.$$

We have

$$\frac{d}{d\alpha} \log \psi(18, \alpha) = \frac{\psi'(18, \alpha)}{\psi(18, \alpha)} = \frac{18\alpha + 2\alpha^3}{1 + 18\alpha^2 + \alpha^4} - \frac{4}{\sqrt{16 + (1-\alpha)^2}} \,.$$

Denote

(10) $$\varphi_1(\alpha) = \frac{18\alpha + 2\alpha^3}{1 + 18\alpha^2 + \alpha^4} \,,$$

(11) $$\varphi_2(\alpha) = \frac{4}{\sqrt{16 + (1-\alpha)^2}} \,.$$

Let us observe that

$$\varphi_1(0) = 0 \,, \quad \varphi(1) = 1 \,.$$

It is easy to verify that

$$\varphi_1'(\alpha) = \frac{-2\alpha^6 - 18\alpha^4 - 318\alpha^2 + 18}{(1 + 18\alpha^2 + \alpha^4)^2}$$

has only one zero $\alpha_1 \in (0, 1)$ where there is a local maximum greater than 1. The function $\varphi_2(\alpha)$ is increasing in the interval $\langle 0, 1 \rangle$ and

$$\varphi_2(0) = \frac{4}{\sqrt{17}} < 1 \,, \quad \varphi_2(1) = 1 \,.$$

Therefore the graphs of the functions φ_1 and φ_2 have only one point $\alpha_2 \in (0, 1)$ in common. For $\alpha \in (0, \alpha_2)$ we have $\varphi_2(\alpha) > \varphi_1(\alpha)$, whereas $\varphi_2(\alpha) < \varphi_1(\alpha)$ for

$\alpha \in (\alpha_2, 1)$. This means that the function $\psi(18, \alpha)$ has a local minimum at the point α_2, which is less than 1.

From the above considerations it follows that the inequality (9) holds for any $\alpha \in (0, 1)$. Finally, for $r \le \sqrt{5} - 2$ and $0 \le \alpha \le 1$ we have

$$\max_{\gamma} |F(\gamma z_0)| \le |F(z_0)| . \tag{12}$$

If $F \in H^*$, then it follows from the formula (2) and the well-known estimates that

$$\frac{|z|(1-|z|)}{(1+|z|)^3} \le |F(z)| \le \frac{|z|(1+|z|)}{(1-|z|)^3} . \tag{13}$$

The estimate (13) is sharp. The equality holds for the function

$$F(z) = \frac{z}{(1-z)^2} \cdot \frac{1+z}{1-z}$$

at the points $z = |z|$ and $z = -|z|$, respectively.

Let us now consider the case when the l.h.s. in (6) is majorized by $\max\limits_{|z|=r^2} |F(z)|$. It follows from the inequality (13) that

$$\frac{\max\limits_{|z|=r^2} |F(z)|}{|F(z_0)|} \le \frac{\dfrac{r^2(1+r^2)}{(1-r^2)^3}}{\dfrac{r(1-r)}{(1+r)^3}} = \frac{r(1+r^2)}{(1-r)^4} \le 1 , \quad |z_0| = r$$

provided that $r \le 2 - \sqrt{3} \approx 0.268$. By Lemma D and (12) we have then

$$|f(z)| \le |F(z)|$$

for $|z| \le \sqrt{5} - 2$.

It is not certain if the result obtained is sharp.

Note. For the pair of functions

$$F(z) = \frac{z}{(1-z)^2} \cdot \frac{1+z}{1-z} \quad \text{and} \quad f(z) = F(z^2)$$

we have

$$\left| \frac{f(-r)}{F(-r)} \right| = \frac{r(1+r^2)}{(1-r)^4} .$$

This means that the inequality

$$|f(z)| \le |F(z)|$$

not always holds for $|z| > 2 - \sqrt{3}$.

We conjecture that the radius of majorization $r_0(\mathcal{H}, H^*)$ is equal $2 - \sqrt{3}$.

REFERENCES

[1] Bogowski, F. ; Bucka, Cz. , *On the relation between subordination and majorisation in a class of holomorphic functions* , Ann. Univ. Mariae Curie–Sklodowska Sect. A, in this volume.

[2] Goluzin, G. M. , *On majorants of subordinate analytic functions* , I. Mat. Sb., 29(71), No 1 (1951), 209–224 (in Russian).

[3] Rogosiński, W. , *Zum Majorantenprinsip der Funktionentheorie* , Math.Zeitschr., 37(1933), 210–236.

[4] Szapiel, W. , *On a theorem of Kirwan* , Zeszyty Nauk. Politech. Rzeszowskiej, Mat. Fiz., 16(1984), 43–58 (in Polish).

STRESZCZENIE

W pracy badana jest zależność między podporządkowaniem a majoryzacją w klasie H^* funkcji prawie gwiaździstych.

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