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An Inversion Formula for the Laplace Transform

Abstract. The purpose of this paper is to show that each right continuous and bounded function $f:[0,\infty)\to\mathbb{R}$ is uniquely determined by its Laplace transform $\mathcal{L}f$. Moreover, we give an inversion formula for the Laplace transform of f.

In the theory of Markov Processes the following theorem is often used:

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a bounded and right continuous function. Then for every $t\in[0,\infty)$ the value $f(t)$ is uniquely determined by the Laplace transform of f.*

In Sharpe's book "General Theory of Markov Processes" ([2], p.17) an explicit formula is given which determines the values of a right continuous function bounded on $[0,\infty)$ by means of its Laplace transform. The purpose of this paper is to show that this formula is incorrect and to give its correct version. Theorem 1 is an immediate consequence of the correct formula. In Section 1 we shortly recall that Theorem 1 can be easily proved without using any explicit formula.

1. A non-constructive proof of Theorem 1. Let $f:[0,\infty)\to\mathbb{R}$ be bounded and Lebesgue measurable. Denote by $\mathcal{L}f$ the Laplace transform of f. Then we have

Theorem 2. *Each continuous and bounded function $f:[0,\infty)\to\mathbb{R}$ is uniquely determined by its transform $\mathcal{L}f$.*

Proof. Let

$$\alpha(t)=\int_0^t f(s)\,ds\,.$$

Then

$$\mathcal{L}f(\lambda)=\int_0^\infty e^{-\lambda t}f(t)\,dt=\int_0^\infty e^{-\lambda t}\,d\alpha(t)\,.$$

Therefore Theorem 2 follows from Theorem 6.3 in [4], p.63, concerning the Laplace–Stjeltjes integral.

Proof of Theorem 1. Observe that, if the function f is right continuous and bounded in $[0,\infty)$, then the function

$$g(t) = e^{-t}\int_0^t f(s)\,ds$$

is continuous and bounded on $[0,\infty)$. Furthermore

$$f(t) = e^t(g(t) + g'_+(t)) ,$$

where g'_+ is the right-hand derivative of g. Therefore it is sufficient to prove that g is uniquely determined by $\mathcal{L}f$. But this is a consequence of the previous theorem because

$$(\mathcal{L}g)(\lambda) = \frac{1}{\lambda+1}(\mathcal{L}f)(\lambda+1) .$$

2. An inversion formula for the Laplace transform of a bounded right continuous function. On the page 17 of M.Sharpe's book [2] an inversion formula (4.14) for Laplace transform of a right continuous and bounded function $f : [0,\infty) \to \mathbb{R}$ is given in the form:

$$f(t) = \lim_{\varepsilon\to 0^+}\lim_{\lambda\to\infty}\sum_{\lambda t<k\leq(\lambda+\varepsilon)t}(-1)^k\frac{\lambda^k}{k!}\phi^{(k)}(\lambda) ,$$

where ϕ denotes the Laplace transform of the function f and $\phi^{(k)}$ the k-th derivative of ϕ. However, the method of the proof is only indicated and the formula is not correct. Indeed, if $f\equiv 1$, then the right hand side of the above equality is equal 0 because

$$\phi(\lambda) = (\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda}$$

$$\phi^{(k)}(\lambda) = (-1)^k\frac{k!}{\lambda^{k+1}}$$

and then

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\lim_{\lambda\to\infty}\sum_{\lambda t<k\leq(\lambda+\varepsilon)t}(-1)^k\frac{\lambda^k}{k!}\phi^{(k)}(\lambda) = \\
&= \lim_{\varepsilon\to 0^+}\lim_{\lambda\to\infty}\sum_{\lambda t<k\leq(\lambda+\varepsilon)t}\frac{1}{\lambda} = \\
&= \lim_{\varepsilon\to 0^+}\lim_{\lambda\to\infty}\frac{[(\lambda+\varepsilon)t]-[\lambda t]}{\lambda} = 0 ,
\end{aligned}$$

because $\frac{\varepsilon t-2}{\lambda} \leq \frac{[(\lambda+\varepsilon)t]-[\lambda t]}{\lambda} \leq \frac{\varepsilon t+2}{\lambda}$.

We are now going to improve the above incorrect fomula by using the same method which is indicated by M.Sharpe. This method (probably originated in the unaccessible papers of Dubourdieu and Feller mentioned in [4], p.295) is based on some lemmas from Feller's book [1]. The first of them is to be found in chapter VII, §1.

Lemma 1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function, continuous at the point $t \in \mathbb{R}$. Let Y_λ, $\lambda > 0$ be a family of non-negative random variables, such that Y_λ has the expected value t and variance σ_λ^2. Denote by F_λ the distribution function of the random variable Y_λ. If $\lim\limits_{\lambda\to\infty} \sigma_\lambda^2 = 0$ then $\lim\limits_{\lambda\to\infty} \int_{-\infty}^{\infty} u(x)\, dF_\lambda(x) = u(t)$.*

An easy proof is based on Chebyshev's inequality and is omitted.

Lemma 2. *For any strictly positive x and t*

$$\lim_{\lambda\to\infty} e^{-\lambda t} \sum_{k\leq\lambda x} \frac{(\lambda t)^k}{k!} = \begin{cases} 0 & , \textit{for } x < t \\ 1 & , \textit{for } x > t \end{cases} .$$

Proof. The proof is based on the well-known argument given in [1] chapter VIII, §6. Let $t \in (0,\infty) \setminus \{x\}$ be fixed. Define the function u by

$$u(y) = \begin{cases} 1 & , \text{for } y < x \, , \\ 0 & , \text{for } y \geq x \, . \end{cases}$$

Let $X_{\lambda t}$ be a random variable having Poisson distribution with the parameter λt. Then for $Y_\lambda = \frac{1}{\lambda} X_{\lambda t}$ we have

$$E[Y_\lambda] = E\big[\frac{1}{\lambda} X_{\lambda t}\big] = \frac{1}{\lambda} E[X_{\lambda t}] = t$$

and

$$\sigma_\lambda^2 = E[(Y_\lambda - t)^2] = E\big[\big(\frac{1}{\lambda} X_{\lambda t} - t\big)^2\big] = \frac{1}{\lambda^2} E[(X_{\lambda t} - \lambda t)^2] = \frac{t}{\lambda} \, .$$

Hence $\sigma_\lambda^2 = \frac{t}{\lambda} \to 0$ as $\lambda \to \infty$. Moreover, the function u is continuous at the point t and

$$\int_{-\infty}^{\infty} u(y)\, dP\big(\frac{1}{\lambda} X_{\lambda t} \leq y\big) = \sum_{k=0}^{\infty} u\big(\frac{k}{\lambda}\big) e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \sum_{k\leq\lambda y} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \, .$$

Hence Lemma 2 is a consequence of Lemma 1.

Theorem 3. *Let $f : [0,\infty) \to \mathbb{R}$ be a right continuous and bounded function. Let $\phi = \mathcal{L}f$ be the Laplace transform of f. Then for every $x \geq 0$ we have*

$$f(x) = \lim_{\epsilon\to 0^+} \lim_{\lambda\to\infty} \frac{1}{\epsilon} \sum_{\lambda x < k \leq \lambda(x+\epsilon)} (-1)^k \frac{\lambda^k}{k!} \phi^{(k)}(\lambda). ,$$

where $\phi^{(k)}$ denotes the k-th derivative ϕ.

Proof. It is sufficient to observe that, as a consequence of the purely analytical Theorem 7d in [4], p.295, we have

$$\lim_{\lambda\to\infty}\sum_{k\le\lambda x}(-1)^k\frac{\lambda^k}{k!}\phi^{(k)}(\lambda)=\int_0^x f(t)\,dt$$

for every $x\in(0,\infty)$. We shall, however, present a probabilistic proof of this formula, based on Lemma 2. Since

$$(-1)^k\phi^{(k)}(\lambda)=\int_0^\infty e^{-\lambda t}t^k f(t)\,dt\,,$$

for any $x\in(0,\infty)$ we have

$$\sum_{k\le\lambda x}e^{-\lambda t}(-1)^k\frac{\lambda^k}{k!}\,\phi^{(k)}(\lambda)=$$

$$=\int_0^{2x+1}\sum_{k\le\lambda x}e^{-\lambda t}\frac{(\lambda t)^k}{k!}f(t)\,dt+\int_{2x+1}^\infty\sum_{k\le\lambda x}e^{-\lambda t}\frac{(\lambda t)^k}{k!}f(t)\,dt\,,$$

where the first integral tends to $\int_0^x f(t)\,dt$ as $\lambda\to\infty$, by the Lebesgue's bounded convergence theorem and by Lemma 2. Therefore it remains to prove that the second one tends to zero as $\lambda\to\infty$. To this end observe that

$$\int_{2x+1}^\infty\sum_{k\le\lambda x}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,|f(t)|\,dt=\int_{2x+1}^\infty P(X_{\lambda t}\le\lambda x)|f(t)|\,dt\le$$

$$\le\sup_{x\in(0,\infty)}|f(x)|\int_{2x+1}^\infty P(X_{\lambda t}\le\lambda x)\,dt\,,$$

where $X_{\lambda t}$ is a Poisson distributed random variable with parameter λt. We are going to complete the proof by showing that

$$\int_{2x+1}^\infty P(X_{\lambda t}\le\lambda x)\,dt\le\frac{48}{\lambda^2}+\frac{8}{\lambda^3}\,.$$

According to the formulas (8.1)–(8.4) in [3], p.90, we have

$$E[X_\eta]=\eta\,,\quad E[X_\eta^2]=\eta^2+\eta\,,\quad E[X_\eta^3]=\eta^3+3\eta^2+\eta\,,$$
$$E[X_\eta^4]=\eta^4+6\eta^3+7\eta^2+\eta\,,$$

and so

$$E[(X_\eta-\eta)^4]=E[X_\eta^4]-4\eta E[X_\eta^3]+6\eta^2E[X_\eta^2]-4\eta^3E[X_\eta]+\eta^4=3\eta^2+\eta\,.$$

By the inequality of Markov we have, for $t\in(2x+1,\infty)$

$$P(X_{\lambda t}\le\lambda x)=P(\lambda t-X_{\lambda t}\ge\lambda t-\lambda x)\le P(|X_{\lambda t}-\lambda t|\ge|\lambda x-\lambda t|)\le$$

$$\le\frac{1}{(\lambda x-\lambda t)^4}\,E[(X_{\lambda t}-\lambda t)^4]\le\frac{2^4}{\lambda^4t^4}\,E[(X_{\lambda t}-\lambda t)^4]\,.$$

Hence

$$\int_{2x+1}^{\infty} P(X_{\lambda t} \le \lambda x)\, dt \le \frac{48}{\lambda^2} \int_1^{\infty} \frac{dt}{t^2} + \frac{16}{\lambda^3} \int_1^{\infty} \frac{dt}{t^3} = \frac{48}{\lambda^2} + \frac{8}{\lambda^3},$$

and the proof is complete.

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