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On Almost Uniform Convergence of Some Sequences of States on Infinite Tensor Products of Von Neumann Algebras

Abstract. Some sequences of states corresponding to the state μ on an infinite tensor product of W^*-algebras converge almost uniformly to μ.

D. Petz in [3] and [4] introduced the notion of almost uniform convergence on the predual of a von Neumann algebra $\mathcal{A}$ as follows.

Let ω be a faithful normal state, and ϕ a normal hermitian functional on $\mathcal{A}$. For any projection p belonging to $\mathcal{A}$, denote by $\|\phi\|_{p,\infty}$ the infimum of nonnegative real numbers λ for which $|\phi(T)| \leq \lambda\omega(T)$ for each $T \in \mathcal{A}$ between $\mathbb{O}$ and $\mathbb{P}$. We say that a sequence ϕ_n of normal hermitian functionals on $\mathcal{A}$ converges to ϕ (also normal hermitian) ω-almost uniformly if, for each $\varepsilon > 0$, there exists a projection p in $\mathcal{A}$ such that $\|\phi_n - \phi\|_{p,\infty} \to 0$ and $\omega(1-p) \leq \varepsilon$.

In [1] and [2] we considered families of normal normed states on finite tensor products of von Neumann algebras. We found necessary and sufficient conditions for the existence of projective limits of such families on infinite local tensor products. Let $\mathcal{A}_i$, $i \in \mathbb{N}$, be a family of von Neumann algebras and α_i an arbitrary fixed normal normed state on $\mathcal{A}_i$. Denote by $\mathcal{A}^n$ the tensor product $\bigotimes_{i=1}^{n} \mathcal{A}_i$ and by $\mathcal{A}$ the local tensor product $\bigotimes_{i=1}^{\infty}(\mathcal{A}_i, \alpha_i)$. Let μ_n be a normal normed state on $\mathcal{A}^n$. We say that the family $\{\mu_n\}$ is consistent if, for any $A \in \mathcal{A}^n$ and $m > n$, $\mu_n(A) = \mu_m(A \bigotimes \mathbf{1}_{n+1,m})$ where $\mathbf{1}_{n+1,m}$ denotes the identity in $\bigotimes_{i=n+1}^{m} \mathcal{A}_i$. The state μ on $\mathcal{A}$ will be called the projective limit of μ_n if $\mu_n(A) = \mu(A \bigotimes \mathbf{1}_{n+1,\infty})$ where $\mathbf{1}_{n+1,\infty}$ is the identity in $\bigotimes_{i=n+1}^{\infty} (\mathcal{A}_i, \alpha_i)$. We proved that the consistent family of states μ_n has the projective limit on $\bigotimes_{i=n+1}^{\infty} (\mathcal{A}_i, \alpha_i)$ if and only if the sequence of states $\nu_n = \mu_n \otimes \bigotimes_{i=n+1}^{\infty} \alpha_i$ converges in norm or weakly (to μ). In [1] we considered a particular case where $\mathcal{A}_i = B(H_i)$, i.e. the algebra of all bounded operators acting in some separable Hilbert space H_i.

The aim of this part of the paper is to show that the sequence ν_n converges in this special case almost uniformly to μ.

Proposition 1. *Let, for each natural i, H_i be a separable Hilbert space, x_i*

some unit vector in H_i and α_i the pure state on $B(H_i)$ generated by x_i. Let ω_i be a faithful normal normed state ob $B(H_i)$ such that the product state $\omega = \bigotimes_{i=1}^{\infty} \omega_i$ exists on $\bigotimes_{i=1}^{\infty}(B(H_i), \alpha_i)$. Assume that $\{\mu_n\}$ is the consistent family of states on finite tensor products $\bigotimes_{i=1}^{\infty} B(H_i)$ with the projective limit μ. Then the sequence of states on $\mathcal{A} = \bigotimes_{i=1}^{\infty}(B(H_i), \alpha_i)$ given by the formula $\nu_n = \mu_n \otimes \bigotimes_{i>n} \alpha_i$ converges ω-almost uniformly to μ.

Proof. Recall that $\mathcal{A}$ is the algebra of all bounded operators on $\bigotimes_{i=1}^{\infty}(H_i, x_i)$. Put

$$p_n = 1_{1,n} \otimes \widehat{x}_{n+1} \otimes \widehat{x}_{n+2} \otimes \ldots$$

where $\widehat{x}_i$ is the projection on the one-dimensional subspace of H_i generated by x_i.

Remark ([1]) that p_n is a well-defined projection in $\mathcal{A}$. We know that the existence of the product of ω_i on $\mathcal{A}$ is equivalent to the convergence of the product of numbers $\omega_i(\widehat{x}_i)$ (see, for example, [1]). So, for each positive ε, we can find a positive integer k such that

$$\omega(p_k) > 1 - \varepsilon .$$

The last inequality is equivalent to $\omega(1 - p_k) < \varepsilon$ because ω is normed.

Let T be a positive operator satisfying $T \le p_k$. It is easy to see that T is of the form

$$T = \widetilde{T} \otimes \widehat{x}_{k+1} \otimes \widehat{x}_{k+2} \otimes \ldots$$

where $\widetilde{T}$ acts in $\bigotimes_{i=1}^{k} H_i$, and $\widetilde{T} \le 1_{1,k}$.

We show that

$$|(\mu - \nu_n)(T)| = (\nu_n - \mu)(T) .$$

Really, we know ([2]) that $\nu_n(T)$ converges to $\mu(T)$. We also have, for $n > k$,

$$\begin{aligned}
\nu_{n+1}(T) &= \nu_{n+1}(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \widehat{x}_{n+1} \otimes \ldots) \\
&= \mu_{n+1}(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_{n+1}) \\
&\le \mu_{n+1}(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes 1_{n+1}) \\
&= \mu_n(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n) \\
&= \nu_n(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \ldots) = \nu_n(T) .
\end{aligned}$$

So, the sequence $\nu_n(T)$ is decreasing and $\nu_n(T) \geq \mu(T)$. Hence

$$\begin{aligned} |(\mu - \nu_n)(T)| &= (\nu_n - \mu)(T) \\ &= (\nu_n - \mu)(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \widehat{x}_{n+1} \otimes \ldots) \\ &\leq (\nu_n - \mu)(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \mathbf{1}_{n+1,\infty}) \\ &= \nu_n(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \mathbf{1}_{n+1,\infty}) \\ &\quad - \mu(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n \otimes \mathbf{1}_{n+1,\infty}) \\ &= \mu_n(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n) \\ &\quad - \mu_n(\widetilde{T} \otimes \widehat{x}_{k+1} \otimes \ldots \otimes \widehat{x}_n) \end{aligned}$$

So, for $n > k$, the norm $\|\mu - \nu_n\|_{p_k,\infty}$ is equal to zero, which ends the proof.

In the general case, we have the following

Proposition 2. *Let μ be the projective limit on $\mathcal{A} = \bigotimes_{i=1}^{\infty}(\mathcal{A}_i, \alpha_i)$ of the consistent family of states μ_n. Suppose that $\mathcal{A}$ is represented as the operator algebra acting on the Hilbert space $H = \bigotimes_{i=1}^{\infty}(H_i, x_i)$. If μ can be written down as $\mu(T) = (Th, h)$ where $h = \sum_{n=1}^{\infty} a_n y_n$ with y_n belonging to the standard basis in H and with a_n such that the series $\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}|a_n|^2\right)^{1/2}$ is convergent, then ν_n converges to μ almost uniformly with respect to every normal faithful state.*

Proof. We have $h = \sum_{n=1}^{\infty} a_n y_n$ where

$$y_n = \sum_{i=1}^{l_n} z_i \otimes \bigotimes_{i>l_n} x_i .$$

For each positive integer k, define the unit vector

$$f_k = \sum_{n=1}^{k} \frac{a_n}{\sum_{n=1}^{k} a_n^2} y_{nk}$$

where

$$y_{nk} = \begin{cases} \sum_{i=1}^{k} z_i & \text{for } k \leq l_n \\ \sum_{i=1}^{l_n} z_i \otimes \sum_{i=l_n+1}^{k} x_i & \text{for } k > l_n \end{cases}$$

and

$$\widetilde{f}_k = \left(\sum_{n=1}^{k} \frac{a_n}{\sum_{n=1}^{k}|a_n|^2} y_{nk}\right) \otimes \bigotimes_{i>k} x_i .$$

We have

$$\|h - \widetilde{f}_k\| = \Big\| \sum_{n=1}^{k} \Big(a_n y_n - \frac{a_n}{\sum_{n=1}^{k} |a_n|^2} y_n \Big) + \sum_{n=k+1}^{\infty} a_n y_n \Big\|$$

$$\le 1 - \Big(\sum_{n=1}^{k} |a_n|^2 \Big)^{-1/2} + \Big(\sum_{n=k+1}^{\infty} |a_n|^2 \Big)^{1/2}$$

$$\le 2 \sum_{n=k+1}^{\infty} |a_n|^2 + \Big(\sum_{n=k+1}^{\infty} |a_n|^2 \Big)^{1/2}.$$

So, the series

$$\sum_{k=1}^{\infty} \|h - \widetilde{f}_k\|$$

is convergent.

By the same considerations as in [2], we calculate that

$$\|\mu - \nu_k\| \le \|\mu - \omega_{\widetilde{f}_k}\| + \|\omega_{f_k} - \mu_k\|$$

and, by the well-known inequality

$$\|\omega_g - \omega_h\| \le 2\|g - h\| ,$$

we have

$$\|\mu - \nu_k\| \le 4\|h - \widetilde{f}_k\| ,$$

(ω_g denotes here the state generated by the vector g, the vectors h and $\widetilde{f}_k$ induce the states μ and $\omega_{\widetilde{f}_k}$ as well as μ_k and ω_{f_k}, respectively).

Hence the series $\sum_{k=1}^{\infty} \|\mu - \nu_k\|$ is convergent, which, by the corollary of Theorem 1 in [4], ends the proof.

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