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The Universal Teichmüller Space of an Oriented Jordan Curve*)

Abstract. The present author constructs a metric space with certain equivalence classes of automorphisms of an oriented Jordan curve Γ on the Riemann sphere which has all the properties of a *boundary model of the universal Teichmüller space*. Its metric is defined without any reference to the families of quasiconformal automorphisms of the domains complementary with respect to Γ. *Complete boundary transformation*, a *norm* and a *metric* for certain equivalence classes of quasicircles are presented in the first and the second part of this paper. The third one contains the statements and basic facts on the boundary model of the universal Teichmüller space.

0. Introduction. The normalized quasisymmetric (qs) functions of $\mathbb{R}$, with a metric obtained by quasiconformal (qc) extensions, provide the most often quoted model of the universal Teichmüller space (UTS) (cf. [5, p.97]). Similar model of the UTS can be defined by using normalized qs automorphisms of the unit circle T (cf. [4]). The metric can also be defined by qc extensions. In both cases the qs constant cannot be used directly to define the Teichmüller metric in same manner as the qc constant. Both models may be called the *mixed models* of the UTS. A more closer look shows that there are some differences between normalized qc automorphisms and normalized qs functions (cf. [3] and [9]). Some of them are not so much natural for boundary values of qc automorphisms.

To remove these obstacles the author has introduced (cf. [9], [10]) a new characterization for the boundary values of all qc automorphisms of the domains complementary with respect to a *Jordan curve* (Jc) Γ in $\overline{\mathbb{C}}$. Automorphisms of Γ, characterized in this way, are called *quasihomographies* (qh) or 1-*dimensional qc mappings*.

This approach permits a new metric which makes the family of all normalized qh automorphisms of an arbitrary circle Γ in $\overline{\mathbb{C}}$ a metric space. This metric is defined without the use of qc extensions to the complementary domains. Moreover, it is fairly natural to call the foregoing metric space the universal Teichmüller space of a given circle Γ in $\overline{\mathbb{C}}$ (cf. [10]).

If a Jc Γ in $\overline{\mathbb{C}}$ is not a circle in $\overline{\mathbb{C}}$ one cannot extend simply the mentioned construction since families of qc automorphisms of the complementary domains D and D^* are not related by conformal reflexion in Γ.

Hence, one should distinguish two classes of automorphisms of a given Jc Γ in $\overline{\mathbb{C}}$ representing the boundary values of qc automorphisms of D and D^*, respectively.

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Given a Jc Γ in $\overline{\mathbb{C}}$, with D and D^* as the complementary domains, one may consider the problem of distinguishing between the domains D and D^*. If Γ is an oriented Jc in the finite complex plane, the correspondence $\Gamma \leftrightarrow (D, D^*)$ is fairly obvious and can be achieved by assuming that the point at infinity is in D^*. This idea doesn't work if Γ is in $\overline{\mathbb{C}}$ which case is of special interest. One overcomes this difficulty by assuming that Γ is on the Riemann sphere, i.e. $\overline{\mathbb{C}}$ equipped with a fixed conformal structure.

Given an oriented Jc Γ on the Riemann sphere one may uniquely associate with Γ the complementary domains D and D^*, defining the *left-hand side* domain by D and the *right-hand side* domain by D^*. Hence, the correspondence $\Gamma \leftrightarrow (D, D^*)$ is unique.

This way one may uniquely associate with an oriented Jc on the Riemann sphere a space T_Γ and call it the *universal Teichmüller space* of Γ.

This yields a *boundary model* of the UTS of an oriented Jc Γ on the Riemann sphere, compatible with qc model (cf. [9], [10] and [11]).

Suppose that Γ in $\overline{\mathbb{C}}$ is an arbitrary Jc, and let D, D^* be its complementary domains. Suppose that $a \in D$ is arbitrary and $z', z'' \in \Gamma$ are arbitrary and distinct points. Consider

$$[z', z'']^a_D = \sin \pi\omega(a, \langle z', z''\rangle; D), \tag{0.1}$$

where $\langle z', z''\rangle$ is an oriented open arc on Γ with end-points z' and z'', ω being the harmonic measure. Suppose that $z_1, z_2, z_3, z_4 \in \Gamma$ is an ordered quadruple of distinct points. Let

$$[z_1, z_2, z_3, z_4]^a_D = \{[z_2, z_3]^a_D [z_1, z_4]^a_D / ([z_1, z_3]^a_D [z_2, z_4]^a_D)\}^{1/2}. \tag{0.2}$$

It is proved in [11] that this expression is constant as a function of $a \in D$. Hence, define

$$[z_1, z_2, z_3, z_4]_D := [z_1, z_2, z_3, z_4]^a_D \quad \text{for any } a \in D, \tag{0.3}$$

and

$$[z_1, z_2, z_3, z_4]_{D^*} := [z_1, z_2, z_3, z_4]^a_{D^*} \quad \text{for any } a \in D^*. \tag{0.3'}$$

Both expressions, defined by (0.3) and (0.3'), are called the *conjugate harmonic cross-ratios* of $z_1, z_2, z_3, z_4 \in \Gamma$.

The mentioned harmonic cross-ratio is a *direct generalization* of the real-valued cross-ratio and an alternative *conformal invariant* with respect to the modulus of quadrilateral. Moreover, this is defined without any use of special functions and carries over properties of the real-valued cross-ratio, expressed conveniently in the form of equalities (cf. [11] and [12]).

Theorem 1. *Given a quadrilateral $D(z_1, z_2, z_3, z_4)$. Let m and t denote its modulus and harmonic cross ratio, respectively. Then*

$$\begin{cases} m = \mu(t) \\ t = \Phi_{1/m}(1/\sqrt{2}) \end{cases} \tag{0.4}$$

Proof. By the Riemann mapping theorem there exists $0 < k < 1$ and a conformal mapping that maps $D(z_1, z_2, z_3, z_4)$ onto $U(-1/k, -1, 1, 1/k)$, where U is the upper half-plane. Then, (cf. [6], p.280])

$$t = [-1/k, -1, 1, 1/k]_U = \frac{2\sqrt{k}}{1+k} = \Phi_2(k) \tag{0.5}$$

and

$$m = \frac{\mathcal{K}(\sqrt{1-k^2})}{2\mathcal{K}(k)} = \frac{1}{2}\mu(k) \tag{0.6}$$

where Φ_K is the Hersch–Pfluger distortion function in the qc version of the Schwarz Lemma (cf. [2], [8] and [9]), and $\frac{\pi}{2}\mu(k)$ stands for the conformal modulus of Δ slit along the real line from 0 to k, $0 < k < 1$ (cf. [Z4]).

Since

$$\Phi_K(r) = \mu^{-1}\left(\frac{1}{K}\mu(r)\right), \quad 0 \le r \le 1, \quad K > 0,$$

substituting $k = \Phi_2^{-1}(t) = \Phi_{1/2}(t)$ into (0.6) one obtains the first identity in (0.4). The second one is a consequence of the first one and the definition of Φ_K.

Let A_Γ denote the family of all sense-preserving automorphisms of Γ. Evidently $(A_\Gamma, \circ)$ is a group with composition as the group operation and that (A_Γ, d_Γ) is a metric space with d_Γ generated by the *chordal-spherical* distance. Moreover the $(A_\Gamma, \circ, d_\Gamma)$ is a metric group (cf. [10]).

Definition 1. Let Γ be an arbitrary Jc in $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. An automorphism $f \in A_\Gamma$ is said to be in the class $A_D(K)$ if

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]_D) \le [f(z_1), f(z_2), f(z_3), f(z_4)]_D \le \Phi_K([z_1, z_2, z_3, z_4]_D) \tag{0.7}$$

holds for each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$ and a given constant $K \ge 1$.

Substituting D^* for D in the previous definition one describes $A_{D^*}(K)$, $K \ge 1$. A function $f \in A_D(K)$ (or $f \in A_{D^*}(K)$) is said to be a K-*quasihomography* or *1-dimensional* K-*qc automorphism of* Γ.

The classes $A_D(K)$ and $A_{D^*}(K)$, $K \ge 1$, are called *conjugate classes of K-qh or of 1-dimensional K-qc automorphisms of* Γ.

The number

$$K(f) = \inf\{K \ge 1 : f \in A_D(K)\} \tag{0.8}$$

is the *qh constant* or, equivalently the *1-dimensional qc constant*. Similarly, one defines $K_{D^*}(f)$ for $f \in A_{D^*}(K)$.

The reason for introducing $A_D(K)$ and $A_{D^*}(K)$ is that these classes represent the boundary values of all K-qc automorphisms of D and D^*, respectively (cf. [11]).

This paper is a continuation of the research presented in [11] in the direction of the UTS theory.

1. Complete boundary transformation. Obviously a conformal mapping between two Jordan domains is determined by its boundary values. Therefore one may say that conformal mappings have the *boundary character*.

Contrariwise, quasiconformal mappings have the *domain character*. Hence, the following considerations are strictly connected with conformal theory and the boundary values of quasiconformal mappings.

Let Γ_i, $i = 1,2,3$, be arbitrary Jc's in $\overline{\mathbb{C}}$, and let A_{Γ_i} denote all sense-preserving automorphisms of Γ_i, $i = 1,2,3$. By D_i and D_i^* one denotes the domains complementary with respect to Γ_i, $i = 1,2,3$. Moreover, let H, H_*, G and G_* be conformal mappings of D_1 onto D_2, D_1^* onto D_2^*, D_2 onto D_3 and D_2^* onto D_3^*, respectively. For every $f_{kl} \in A_{\Gamma_1}$, $k,l = 1,2$, consider the transformation $\mathbf{S}_{H,H_*}$, described by

(1.1)
$$\mathbf{S}_{H,H_*}\left(\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\right) = (S^{kl}_{H,H_*}(f_{kl})) = \begin{pmatrix} H \circ f_{11} \circ H^{-1} & H \circ f_{12} \circ H_*^{-1} \\ H_* \circ f_{21} \circ H^{-1} & H_* \circ f_{22} \circ H_*^{-1} \end{pmatrix}$$

Since conformal mappings between Jordan domains can be considered as homeomorphisms of their closures , the transformation $\mathbf{S}_{H,H_*}$ is well-defined and called the *complete boundary transformation* mapping $A^{\mathcal{A}}_{\Gamma_1}$ onto $A^{\mathcal{A}}_{\Gamma_2}$, where $\mathcal{A} = \{1,2,3,4\}$.

It is evident that

$$\mathbf{S}_{G \circ H, G_* \circ H_*} = \mathbf{S}_{G,G_*} \circ \mathbf{S}_{H,H_*}, \tag{1.2}$$

which yields

$$\mathbf{S}_{H^{-1},H_*^{-1}} = \mathbf{S}^{-1}_{H,H_*}. \tag{1.3}$$

Write,

$$\tilde{f} = \begin{pmatrix} f & f \\ f & f \end{pmatrix}, \qquad f \in A_{\Gamma_1},$$

and let G_{Γ_1} be the collection of all such elements. Given $\tilde{f}$ and $\tilde{g}$ from G_{Γ_1}, set

$$\tilde{f} * \tilde{g} = \begin{pmatrix} f \circ g & f \circ g \\ f \circ g & f \circ g \end{pmatrix}. \tag{1.4}$$

Hence, $(G_{\Gamma_1}, *)$ is a group and

$$\mathbf{S}_{H,H_*}(\tilde{f} * \tilde{g}) = \mathbf{S}_{H,H_*}(\tilde{f}) * \mathbf{S}_{H,H_*}(\tilde{g}) \tag{1.5}$$

holds for every $\tilde{f}$, $\tilde{g}$ from G_{Γ_1}. One has proved

Theorem 2. *The complete boundary transformation $\mathbf{S}_{H,H_*}$ is an isomorphism between $(G_{\Gamma_1}, *)$ and $(G_{\Gamma_2}, *)$.*

Let

$$S_H = S^{11}_{H,H_*}, \quad S_{H_*} = S^{22}_{H,H_*}, \quad D_{HH^*} = S^{12}_{H,H_*} \quad D_{H_* H} = S^{21}_{H,H_*}. \tag{1.6}$$

Denote by

$$R_{\Gamma_1,\Gamma_2} = H \circ H_*^{-1} \tag{1.7}$$

and call the *conformal representation* of Γ_1 *with respect to* Γ_2. Let $R_\Gamma = R_{\Gamma,T}$ for an arbitrary Jc Γ in $\overline{\mathbb{C}}$. Recall, that, R_Γ is known as the *sewing automorphism* or the *conformal representation* of Γ (cf. [7]).

One obtains the following

Theorem 3. *For an arbitrary Jc* Γ_1, Γ_2 *in* $\overline{\mathbb{C}}$, *and each fixed conformal* H *and* H_*, *as described above, the solution of*

$$S_H(f) = S_{H_*}(f) \tag{1.8}$$

contains infinitely many automorphisms of the form $(H_*^{-1} \circ H)^{\underline{n}}$, *where* $f^{\underline{n}}$ *means n-fold composition of f and* $f^{-\underline{n}} = (f^{-1})^{\underline{n}}$, $n = 0, \pm 1, \pm 2, \ldots$

Proof. It follows immediately, by checking, that $H_*^{-1} \circ H$ satisfies (1.8) and then by using (1.2) and (1.3).

The transformation

$$J_{HH_*} = S_{H_*} \circ S_H^{-1} \tag{1.9}$$

is a self-isomorphism of A_{Γ_2}.

It is obvious that all solutions of (1.8) form a group with composition as the group operation. Denote by $R^{\infty}_{\Gamma_2,\Gamma_1}$ the family of all solutions obtained in Theorem 1, and set

$$R^{\infty}_{\Gamma_1,\Gamma_2} = S_H(R^{\infty}_{\Gamma_2,\Gamma_1}) = S_{H^*}(R^{\infty}_{\Gamma_2,\Gamma_1}).$$

Hence, one may easily see that $(R^{\infty}_{\Gamma_1,\Gamma_2}, \circ)$ is a group generated by R_{Γ_1,Γ_2} and call it the *fix-points group* of J_{HH_*}.

Not without justifications one may call J_{HH_*} the *conjugation operator* in A_{Γ_2}, whose the "real line" consists $R^{\infty}_{\Gamma_1,\Gamma_2}$. It is very probably that $R^{\infty}_{\Gamma_2,\Gamma_1}$ are the only solutions of (1.8). It is obvious that (cf. [11, Theorem 6])

$$K_{D_2}(H \circ H_*^{-1}) = K_{D_2}(H_* \circ H^{-1}), \tag{1.10}$$

and

$$K_{D_2^*}(H \circ H_*^{-1}) = K_{D_2^*}(H_* \circ H^{-1}). \tag{1.10'}$$

Moreover, each of these expressions is a constant, considered as a function of H and H_*.

Let now

$$A^{\infty}_{D_i} = \bigcup_{K \geq 1} A_{D_i}(K) \quad \text{and} \quad A^{\infty}_{D_i^*} = \bigcup_{K \geq 1} A_{D_i^*}(K)$$

for $i = 1, 2, 3$. Obviously, $(A^{\infty}_{D_i}, \circ)$ and $(A^{\infty}_{D_i^*}, \circ)$ are subgroups of $(A_{\Gamma_i}, \circ)$ for $i = 1, 2, 3$.

Hence,

Theorem 4. *Let Γ_1 and Γ_2 be arbitrary Jc's in $\overline{\mathbb{C}}$, and D_1, D_1^*, D_2, D_2^* denote the domains complementary to Γ_1 and Γ_2, respectively. If H and H_* are given conformal mappings of D_1 and D_1^* onto D_2 and D_2^*, then:*

(i) *the transformations S_H and S_{H_*}, defined by (1.6), are homeomorphisms between $(A_{\Gamma_1}, \rho_{\Gamma_1})$ and $(A_{\Gamma_2}, \rho_{\Gamma_2})$, and isomorphisms between $(A_{\Gamma_1}, \circ)$ and $(A_{\Gamma_1}, \circ)$;*

(ii) *S_H is an isomorphism between $(A_{D_1}^{\infty}, \circ)$ and $(A_{D_2}^{\infty}, \circ)$, whereas S_{H_*} is an isomorphism between $(A_{D_1^*}^{\infty}, \circ)$ and $(A_{D_2^*}^{\infty}, \circ)$, such that*

$$S_H(A_{D_1}(K)) = A_{D_2}(K) \quad \text{and} \quad S_{H_*}(A_{D_1^*}(K)) = A_{D_2^*}(K), \quad \text{for} \quad K \geq 1;$$

(iii) *the transformations D_{HH_*} and D_{H_*H} are homeomorphisms between $(A_{\Gamma_1}, d_{\Gamma_1})$ and $(A_{\Gamma_2}, d_{\Gamma_2})$. Moreover,*

$$D_{HH_*}(f) = S_H(f) \circ R = R \circ S_{H_*}(f);$$
$$D_{H_*H}(f) = R^{-1} \circ S_H(f) = S_{H_*}(f) \circ R^{-1};$$

and

$$(D_{HH_*}(f))^{-1} = D_{H_*H}(f^{-1})$$

holds for every $f \in A_{\Gamma_1}$, where $R = R_{\Gamma_1\Gamma_2}$ for shortness.

(iv) *the transformation J_{HH_*}, defined by (1.9), is an automorphism of the metric group $(A_{\Gamma_2}, \circ, \rho_{\Gamma_2})$ and $R_{\Gamma_1\Gamma_2}^{\infty}$ are fix-points of this transformation. Moreover,*

$$S_H = S_R \circ S_{H_*} \quad \text{and} \quad J_{HH_*} = S_R^{-1},$$

where $S_R : A_{\Gamma_2} \to A_{\Gamma_2}$ is defined in analogy to S_H.

2. A norm and a metric for quasicircles.

A K-quasicircle in $\overline{\mathbb{C}}$ is the image of a circle (say, the unit circle) under a K-qc mapping of $\overline{\mathbb{C}}$. Quasiconformal mappings preserve sets of zero measure, so every quasicircle is of zero area. On the other hand, a quasicircle need not be rectifiable. Moreover, the Hausdorff dimension of a quasicircle may take any value from [1;2) (see [2]). A considerable amount of the main properties of quasicircles may be found in [1].

Assume that $\Gamma_1 = T$ and Γ_2 is denoted by Γ. According to this let $D_1 = \Delta$, $D_1^* = \Delta^*$, $D_2 = D$ and $D_2^* = D^*$. Recall of [10], that $A_\Delta(K) = A_{\Delta^*}(K)$ for every $K \geq 1$. It is easily seen that

$$K_D(H_* \circ H^{-1}) = K_{D^*}(H_* \circ H^{-1}). \tag{2.1}$$

This identity, together with identities described by (1.10) and (1.10′), implies the following

Definition 2. Let Γ be an arbitrary Jc in $\overline{\mathbb{C}}$. The common value described by (2.1), (1.10) and (1.10') denoted is by K_Γ.

Two Jc's Γ_1 and Γ_2 in $\overline{\mathbb{C}}$ are said to be equivalent ($\Gamma_1 \sim \Gamma_2$) if there is a homography M such that $\Gamma_2 = M(\Gamma_1)$. If $\Gamma_1 \sim \Gamma_2$ then $K_{\Gamma_1} = K_{\Gamma_2}$. Let $\mathbf{\Gamma}$ be the family of all Jc's in $\overline{\mathbb{C}}$, and let

$$\mathbf{\Gamma}_1 = \mathbf{\Gamma}/\sim \tag{2.2}$$

be the space of the equivalence classes whose elements are denoted by $[\Gamma]$. Identifying every element of $\mathbf{\Gamma}_1$ with its normalized conformal representation on the unit circle, one notes that this makes $\mathbf{\Gamma}_1$ a group.

Hence,

Definition 3. For each $[\Gamma] \in \mathbf{\Gamma}_1$, the value

$$\|[\Gamma]\| = \tfrac{1}{2}\log K_\Gamma \tag{2.3}$$

is called a *norm* in $\mathbf{\Gamma}_1$.

Let $\mathbf{\Gamma}^\infty$ denote the family of all Jc's Γ in $\overline{\mathbb{C}}$ with finite value of K_Γ. As shown in [Z3, Theorem 10], a Jc $\Gamma \subset \mathbf{\Gamma}^\infty$ if, and only if, Γ is a quasicircle.

Let $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}^\infty$ be arbitrary. The expression

$$q(\Gamma_1, \Gamma_2) = \tfrac{1}{2}\left|\log \frac{K_{\Gamma_1}}{K_{\Gamma_2}}\right| \tag{2.4}$$

is a *pseudometric* in $\mathbf{\Gamma}^\infty$.

To make q a metric one has to introduce a much weaker equivalence relation to $\mathbf{\Gamma}^\infty$. First, one introduces it to $\mathbf{\Gamma}$ by saying that two Jc's Γ_1 and Γ_2 are w-equivalent ($\Gamma_1 \approx \Gamma_2$) if $K_{\Gamma_1} = K_{\Gamma_2}$. Then let

$$\mathbf{\Gamma}_2 = \mathbf{\Gamma}/\approx$$

be the space of the equivalence classes, whose elements are denoted by $\big[[\Gamma]\big]$. Moreover, let $\mathbf{\Gamma}_2^\infty = \mathbf{\Gamma}^\infty/\approx$. Putting

$$q^*\big([[\Gamma_1]], [[\Gamma_2]]\big) = q(\Gamma_1, \Gamma_2)$$

one obtains

Theorem 5. *The* $(\mathbf{\Gamma}_2^\infty, q^*)$ *is a metric space.*

3. Universal Teichmüller space of an oriented Jordan curve. By way of supplementing the investigation of [10] suppose that Γ is an oriented Jc on the Riemann sphere and D, D^* are the left and right-hand side domains, respectively. Set $\mathbf{A}_\Gamma = A_\Gamma \times A_\Gamma$. For arbitrary $f = (f_1, f_2)$ and $g = (g_1, g_2)$ from $\mathbf{A}_\Gamma$, set

$$f \circ g = (f_1, f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2). \tag{3.1}$$

Then, $(\mathbf{A}_\Gamma, \circ)$ is a group with composition as the group operation. Introducing

$$\mathbf{d}_\Gamma(f, g) = \max\{d_\Gamma(f_1, g_1), d_\Gamma(f_2, g_2)\} \tag{3.2}$$

one makes $(\mathbf{A}_\Gamma, \circ, \mathbf{d}_\Gamma)$ a metric group (cf. [Z2]). Moreover, one defines

(3.3) $$A_D^\infty = \bigcup_{K\geq 1} A_D(K) \quad \text{and} \quad A_{D^*}^\infty = \bigcup_{K\geq 1} A_{D^*}(K)$$

then,

(3.4) $$\mathbf{A}_\Gamma(K) = A_D(K) \times A_{D^*}(K) \quad \text{and} \quad \mathbf{A}_\Gamma^\infty = \bigcup_{K\geq 1} \mathbf{A}_\Gamma(K)$$

for $K \geq 1$.

Hence, $(\mathbf{A}_\Gamma^\infty, \circ)$ is a group, as well. Note that each element of $\mathbf{A}_\Gamma^\infty$ is an automorphism of $\Gamma \times \Gamma$.

Two automorphisms $f_1, g_1 \in A_D^\infty$ are said to be equivalent ($f_1 \sim g_1$) if $f_1 \circ g_1^{-1} \in A_D(1)$. Similarly, two automorphisms $f_2, g_2 \in A_{D^*}^\infty$ are equivalent ($f_2 \backsim g_2$) if $f_2 \circ g_2^{-1} \in A_{D^*}(1)$. Introducing

$$T_D = A_D^\infty / \sim \quad \text{and} \quad T_{D^*} = A_{D^*} / \backsim$$

we may call

(3.5) $$T_\Gamma = (T_D, T_{D^*})$$

the *universal Teichmüller space of* Γ.

For an arbitrary $f = (f_1, f_2)$ and $g = (g_1, g_2)$ from $\mathbf{A}_\Gamma^\infty$, one defines

(3.6) $$\begin{aligned} \tau_D(f_1, g_1) &= \tfrac{1}{2} \log K_D(f_1 \circ g_1^{-1}), \\ \tau_{D^*}(f_2, g_2) &= \tfrac{1}{2} \log K_{D^*}(f_2 \circ g_2^{-1}), \end{aligned}$$

and then

(3.7) $$\tau_\Gamma(f, g) = \frac{1}{2}(\tau_D(f_1, g_1) + \tau_{D^*}(f_2, g_2)).$$

This is a pseudometric in $\mathbf{A}_\Gamma^\infty$, and $0 \leq \tau_\Gamma(f, g) \leq \log K$ for arbitrary $f, g \in \mathbf{A}_\Gamma(K)$.

Theorem 6. *For an oriented Jc* Γ *on the Riemann sphere and every* $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathbf{A}_\Gamma^\infty$ *implies that:*

(i) $\tau_\Gamma(f, g) = 0$ *if and only if* $f \circ g^{-1} \in \mathbf{A}_\Gamma(1)$;

(ii) $\mathbf{d}_{(-\Gamma)}(f, g) = \mathbf{d}_\Gamma(f, g)$;

(iii) $\tau_{(-\Gamma)}(f, g) \leq \tau_\Gamma(f, g) + 2 \log Q$ *if* Γ *is a* Q*-quasicircle, where* $(-\Gamma)$ *is the Jc obtained from* Γ *by reversing the orientation.*

Proof. The identity

$$\tau_\Gamma(f, g) = 0$$

is equivalent to

$$\tau_D(f_1, g_1) = 0 \quad \text{and} \quad \tau_{D^*}(f_2, g_2) = 0$$

then to

$$f_1 \circ g_1^{-1} \in A_D(1) \quad \text{and} \quad f_2 \circ g_2^{-1} \in A_{D^*}(1)$$

and to

$$f \circ g^{-1} = (f_1 \circ g_1^{-1}, f_2 \circ g_2^{-1}) \in \mathbf{A}_\Gamma(1).$$

Obviously,

$$\mathbf{A}_\Gamma = \mathbf{A}_{(-\Gamma)}$$

and, because of (3.2), (ii) follows. If Γ is a Q-quasicircle then, by [Z3, Theorem 11], there exist $1 \leq L_1, L_2 \leq Q^4 K$, such that the following inclusions

$$A_D(K) \subset A_{D^*}(L_1) \quad \text{and} \quad A_{D^*}(K) \subset A_D(L_2) \tag{3.8}$$

hold for every $K \geq 1$. Then, (iii) follows by elementary calculation.

If Γ is a circle then $Q = 1$ and, because of (iii), one obtains

$$\tau_{(-\Gamma)}(f,g) = \tau_\Gamma(f,g). \tag{3.9}$$

As a completion of Theorem 6, one has

Corollary 1. *It follows from the above that*

(i) $\mathbf{A}_{(-\Gamma)} = \mathbf{A}_\Gamma$ *always;*

(ii) $\mathbf{A}^\infty_{(-\Gamma)} = \mathbf{A}^\infty_\Gamma$ *if Γ is a quasicircle in $\overline{\mathbb{C}}$;*

(iii) $T_{(-\Gamma)} = T_\Gamma$ *if and only if Γ is a circle in $\overline{\mathbb{C}}$.*

Proof. (i) is obvious, whereas (ii) follows by [11, Theorem 11]. (iii) is a result of the observation that $[f] \in T_\Gamma$ inherits the group structure if and only if $K_D(f) = K_{D^*}(f) = 1$ (cf. [11, Theorem 10]).

Assume now that $[f^1]$, $[f^2]$, $[g^1]$ and $[g^2]$ are elements of T_Γ. Hence

$$[f^1] = [f^2] \iff f^1 \sim f^2 \iff f^1 = h^1 \circ f^2,$$
$$[g^1] = [g^2] \iff g^1 \sim g^2 \iff g^1 = h^2 \circ f^2,$$

where h^1 and h^2 are elements of $A_D(1)$ and $A_{D^*}(1)$, respectively. By this

$$f^1 = (f^1_1, f^1_2) = (h^1_1 \circ f^2_1, h^1_2 \circ f^2_2)$$

and

$$g^1 = (g^1_1, g^1_2) = (h^1_1 \circ g^2_1, h^2_2 \circ g^2_2).$$

Hence

(3.10)

$$\tau_\Gamma(f^1, g^1) = \frac{1}{4} \log K_D(h^1_1 \circ f^2_1 \circ (g^2_1)^{-1} \circ (h^2_1)^{-1}) K_{D^*}(h^1_2 \circ f^2_2 \circ (g^2_2)^{-1} \circ (h^2_2)^{-1})$$
$$= \frac{1}{4} \log K_D(f^2_1 \circ (g^2_1)^{-1}) K_{D^*}(f^2_2 \circ (g^2_2)^{-1}) = \tau_\Gamma(f_2, g_2).$$

It follows from the above considerations that

(3.11) $$\tau_\Gamma^*([f],[g]) = \tau_\Gamma(f,g).$$

Evidently, τ_Γ^* is well-defined and independent of the representation. Thus

Theorem 7. *For an oriented Jc Γ on the Riemann sphere $(T_\Gamma, \tau_\Gamma^*)$ is a metric space.*

Suppose now that $[f] = [g]$. Then $f_1 \sim g_1$ and $f_2 \sim g_2$. Let H and H_* map, as usual, Δ and Δ^* onto D and D^*, respectively. Define

$$\tilde{f}_2 = J_{HH_*}(f_1) \quad \text{and} \quad \tilde{g}_2 = J_{HH_*}(g_1).$$

Then

(3.12) $$\tilde{f}_2 \circ (\tilde{g}_2)^{-1} = J_{HH_*}\big(f_1 \circ (g_1)^{-1}\big)$$

and, because of (iii) of Theorem 4, $\tilde{f}_2 \circ (\tilde{g}_2)^{-1} \in A_{D^*}(1)$. Moreover, let

$$\tilde{f}_1 = J_{H_*H}(f_2) \quad \text{and} \quad \tilde{g}_1 = J_{H_*H}(g_2).$$

Then

(3.12′) $$\tilde{f}_1 \circ (\tilde{g}_1)^{-1} = J_{H_*H}\big(f_2 \circ (g_2)^{-1}\big).$$

Hence, by (iii) of Theorem 4, one can see that $\tilde{f}_1 \circ (\tilde{g}_1)^{-1} \in A_D(1)$.

Introduce

(3.12) $$\mathbf{J}_{HH_*}(f) = \big(J_{H_*H}(f_2), J_{HH_*}(f_1)\big)$$

and, by the above considerations,

(3.13) $$\mathbf{J}_{HH_*}^*([f]) = [\mathbf{J}_{HH_*}(f)].$$

This is a well-defined transformation of T_Γ onto $T_{(-\Gamma)}$.

Then we have

Theorem 8. *For an oriented Jc Γ on the Riemann sphere the following holds:*

(i) *the transformation $\mathbf{J}_{HH_*}$, defined by (3.13), is an automorphism of the metric group $(\mathbf{A}_\Gamma, \circ, d_\Gamma)$, and $R_\Gamma^\infty \times R_\Gamma^\infty$ are fix-points of this transformation;*

(ii) *$(\mathbf{J}_{HH_*})^{-1} = \mathbf{J}_{H_*H}$, and $\mathbf{J}_{HH_*}$ is an involution of $\mathbf{A}_\Gamma^\infty$, provided Γ is a quasicircle;*

(iii) *$\mathbf{J}_{HH_*}^*$ is an isometry between $(T_\Gamma, \tau_\Gamma^*)$ and $(T_{(-\Gamma)}, \tau_{(-\Gamma)}^*)$.*

Proof. (i) follows from Corollary 1. The identity of (ii) is a simple consequence of the definition of $\mathbf{J}_{HH_*}$. (iii) follows by the definition of τ_Γ^*. The identity (ii) of Corollary 1 and the identity in (ii) of this theorem imply the second statement of (ii).

By (iii) and (iv) of Theorem 4 and [11, Theorem 11], it follows that D_{HH_*} and D_{H_*H} map $A_T(K)$ into $A_D(K \cdot K_\Gamma)$ or into $A_{D^*}(K \cdot K_\Gamma)$, $K \geq 1$. If, in addition, Γ is a quasicircle in $\overline{\mathbb{C}}$ then both the transformations map A_T^∞ onto $A_D^\infty = A_{D^*}^\infty$ (cf. [11, Theorem 11]).

In order to lift D_{HH_*} and D_{H_*H} on the respective UTS, suppose that $f, g \in A_T^\infty$ are such that $f \circ g^{-1} \in A_T(1)$. Then,

$$D_{HH_*}(f) \circ (D_{HH_*}(g))^{-1} = S_H(f \circ g^{-1}) \in A_D(1) \tag{3.14}$$

and

$$D_{H_*H}(f) \circ (D_{H_*H}(g))^{-1} = S_{H_*}(f \circ g^{-1}) \in A_{D^*}(1). \tag{3.14'}$$

According to the previous cases, one defines $D^*_{HH_*}$ and $D^*_{H_*H}$, that map $T_T = T_\Delta = T_{\Delta^*}$ onto T_D and T_{D^*}, respectively. Hence, the notion

$$\mathbf{S}^*_{H,H_*} := \begin{pmatrix} S^*_{H_*} & D^*_{HH_*} \\ D^*_{H_*H} & S^*_{H_*} \end{pmatrix}. \tag{3.15}$$

which means that S^*_{H,H_*} is defined by matrix representation.

Then, for sake a portion of simplification, one writes

$$\mathbf{S}^*_{H,H_*} : \begin{pmatrix} T_T & T_T \\ T_T & T_T \end{pmatrix} \to \begin{pmatrix} T_D & T_D \\ T_{D^*} & T_{D^*} \end{pmatrix}. \tag{3.16}$$

Let, as before, Γ_1 and Γ_2 be arbitrary oriented Jc's on the Riemann sphere, and let H, H_* and G, G_* be conformal mappings of Δ and Δ_* onto D_1, D_1^* and D_2, D_2^*, respectively. Consider the following transformation

$$\mathbf{J}_{\Gamma_1\Gamma_2}(f) = \left(J_{H_*G}(f_2), J_{HG_*}(f_1)\right) \tag{3.17}$$

where

$$J_{H_*G} = S_G \circ S_{H_*}^{-1} \quad \text{and} \quad J_{HG_*} = S_{G_*} \circ S_H^{-1} \tag{3.18}$$

map $\mathbf{A}^\infty_{\Gamma_1}$ onto $\mathbf{A}^\infty_{\Gamma_2}$. By arguments related to those that we used while defining τ^* and $\mathbf{J}^*_{HH_*}$, one states

$$\mathbf{J}^*_{\Gamma_1\Gamma_2}([f]) = [\mathbf{J}_{\Gamma_1\Gamma_2}(f)]. \tag{3.19}$$

Then

Remark 1. If $\Gamma_1 = \Gamma_2 = \Gamma$, one may identify G and G_* with H and H_*, respectively, and set

$$\mathbf{J}_\Gamma = \mathbf{J}_{\Gamma\Gamma} = \mathbf{J}_{HH_*}. \tag{3.20}$$

Henceforth, the notion $\mathbf{J}_\Gamma$ can be used instead of $\mathbf{J}_{HH_*}$, as more adequate in these circumstances (cf. [13]).

Remark 2. By the previous arguments, one defines

$$\mathbf{J}^*_\Gamma = \mathbf{J}^*_{\Gamma\Gamma}. \tag{3.21}$$

Hence,

Theorem 9. *For oriented Jc's Γ_1 and Γ_2 on the Riemann sphere the following holds:*

(i) $\mathbf{J}_{\Gamma_1\Gamma_2}$ *is an isomorphisms between* $(\mathbf{A}^\infty_{\Gamma_1}, \circ)$ *and* $(\mathbf{A}^\infty_{(-\Gamma_2)}, \circ)$;

(ii) $\mathbf{J}^*_{\Gamma_1\Gamma_2}$ *is an isometry between* $(T_{\Gamma_1}, \tau^*_{\Gamma_1})$ *and* $(T_{(-\Gamma_2)}, \tau^*_{(-\Gamma_2)})$;

(iii) $(\mathbf{J}_{\Gamma_1\Gamma_2})^{-1} = \mathbf{J}_{\Gamma_2\Gamma_1}$ *and* $(\mathbf{J}^*_{\Gamma_1\Gamma_2})^{-1} = \mathbf{J}^*_{\Gamma_2\Gamma_1}$;

Proof. The condition (i) is a simple consequence of the respective condition in Theorem 2. Since $\mathbf{J}_{\Gamma_1\Gamma_2}$ preserves the qh constant (ii) follows by an easy calculation similar to those that were used before. (iii) can be check, immediately.

Remark 3. If Γ is a circle in $\overline{\mathbb{C}}$, then $A_D(K) = A_{D^*}(K)$ for every $K \geq 1$. Hence, $\mathbf{A}_\Gamma(K) = A_D(K) \times A_{D^*}(K)$ can be identified with $A_\Gamma(K)$ on Γ (see [9]). Since $K_D(f) = K_{D^*}(f)$ if and only if Γ is a circle in $\overline{\mathbb{C}}$, τ^*_Γ is isometric with τ^*_Γ. By this one may identify $(T_\Gamma, \tau^*_\Gamma)$ with $(T_\Gamma, \tau^*_\Gamma)$ defined in [10].

Suppose that Γ_1 and Γ_2 are oriented Jc's on the Riemann sphere with D_1, D_1^* and D_2, D_2^*, as the respective left and right-hand side domains. Let H and H_* map conformaly D_1 and D_1^* onto D_2 and D_2^*, respectively. One may then consider the *parallel transformation*

$$\mathbf{S}_{\Gamma_1\Gamma_2} = (S_H, S_{H_*}) \tag{3.22}$$

and then

$$\mathbf{S}^*_{\Gamma_1\Gamma_2} = (S^*_H, S^*_{H_*}) \tag{3.23}$$

that map $\mathbf{A}^\infty_{\Gamma_1}$ and T_{Γ_1} onto $\mathbf{A}^\infty_{\Gamma_2}$ and T_{Γ_2}, respectively.

Further development of the ideas presented in this Chapter, including the case when Γ is an oriented Jc, or an arc of an oriented Jc, on a closed Riemann surface, will be presented in [13] and [14].

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