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Dominik SZYNAL (Lublin)

On Complete Convergence for some Classes of Dependent Random Variables

Abstract. We show that Hsu and Robbins law of large numbers holds for quadruplewise independent random variables but it does not hold for pairwise independent random variables.

1. Introduction and preliminaries. In [5] it was proved that Kolmogorov's strong law of large numbers can be extended to pairwise independent identically distributed random variables. We note that in general Hsu and Robbins law of large numbers is not satisfied for pairwise independent random variables. Speaking more precisely, if $\{X_k, k \geq 1\}$ is a sequence of pairwise independent random variables with $EX_1 = 0$, $EX_1^2 < \infty$, then in general the series $\sum_{n=1}^{\infty} P[|S_n| \geq n\epsilon]$ does not converge for any given $\epsilon > 0$.

Example. Let $\{X_j \, , \, j \geq 1\}$ be a sequence defined as follows

$$X_j = \cos 2\pi(U + (j-1)V) \, , \quad j \geq 1 \, ,$$

where U and V are independent and uniformly distributed random variables on $[0, 1]$.

One can show that X_j, $j \geq 1$, are pairwise independent (cf. [6]) with $EX_j = 0$, $j \geq 1$. Moreover, we see that

$$S_n = \sum_{j=1}^{n} X_j = \frac{\sin \pi n V \cos \pi(2U + (n-1)V)}{\sin \pi V} \, .$$

Note that for any given $\epsilon > 0$ (we assume that $\epsilon < \frac{1}{4}$) we have

$$P[|S_n| > n\epsilon] = \mu_L \left\{ (x,y) \in [0,1] \times [0,1] : \left| \frac{\sin \pi n x}{\sin \pi x} \right| |\cos \pi((n-1)x + 2y)| > \epsilon n \right\} ,$$

where μ_L denotes the Lebesque measure. Taking into account that

$$x - \frac{x^3}{3!} < \sin x < x \, , \, \sqrt{\frac{3}{\pi}} < 1 < \frac{\pi}{3} \, , \, \frac{\pi}{3} - \sqrt{\frac{3}{\pi}} > 0 \, ,$$

we get

$$
\begin{aligned}
&P[|S_n| \geq n\epsilon] \\
&\geq \mu_L \left\{ (x,y) \in [0, \frac{\sqrt{3}}{n\pi}] \times [0,1] : \frac{\sin \pi n x}{\pi x} |\cos \pi((n-1)x + 2y)| \geq n\epsilon \right\} \\
&= \mu_L \left\{ (x,y) \in [0, \frac{\sqrt{3}}{n\pi}] \times [0,1] : n(1 - \frac{\pi^2 x^2 n^2}{6}) |\cos \pi((n-1)x + 2y)| \geq n\epsilon \right\} \\
&\geq \mu_L \left\{ x \in (0, \frac{\sqrt{3}}{n\pi}), y \in (0,1) : |\cos \pi((n-1)x + 2y)| \geq 2\epsilon \right\} \\
&\geq \mu_L \left\{ x \in (0, \frac{1}{n\pi}\sqrt{\frac{3}{\pi}}), y \in (0, (\frac{\pi}{3} - \sqrt{\frac{3}{\pi}})\frac{1}{2\pi})) : |\cos \pi((n-1)x + 2y)| \geq 2\epsilon \right\}
\end{aligned}
$$

But for $x \in (0, \sqrt{3/\pi}(n\pi)^{-1})$, $y \in (0, (\pi/3 - \sqrt{3/\pi})(2\pi)^{-1})$ we have

$$0 < \pi(n-1)x + 2\pi y) < \sqrt{\frac{3}{\pi}} \frac{n-1}{n} + \frac{\pi}{3} - \sqrt{\frac{3}{\pi}} < \frac{\pi}{3} \,,$$

hence

$$\cos 0 \geq \cos(\pi(n-1)x + 2\pi y) \geq \cos \frac{\pi}{3} \,.$$

Therefore for $\epsilon < \frac{1}{4}$

$$
\begin{aligned}
P[|S_n| \geq n\epsilon] &\geq \mu_L \left\{ x \in (0, \frac{1}{n\pi}\sqrt{\frac{3}{\pi}}),\ y \in (0, (\frac{\pi}{3} - \sqrt{\frac{3}{\pi}})\frac{1}{2\pi}) \right\} \\
&= \sqrt{\frac{3}{\pi}} (\frac{\pi}{3} - \sqrt{\frac{3}{\pi}}) \frac{1}{2\pi^2 n}
\end{aligned}
$$

which implies that

$$\sum_{n=1}^{\infty} P[|S_n| \geq n\epsilon] = \infty \,,$$

i.e. $S_n/n \not\to 0$ completely as $n \to \infty$ (Hsu and Robbins law of large numbers does not hold).

Let $\{X_k\,, k \geq 1\}$ be a sequence of independent identically distributed random variables with a finite expectation μ. Put $S_n = \sum_{k=1}^{n} X_k$ and for any given $\epsilon > 0$ define $A_n = [|S_n - n\mu| \geq n\epsilon]$. The strong law of large numbers can be formulated in the form $N(\epsilon, \infty) := \sum_{n=1}^{\infty} I[A_n] < \infty$ a.s. for all $\epsilon > 0$. Hsu and Robbins [7] proved that

$$\text{(1)} \qquad EN(\epsilon, \infty) = \sum_{n=1}^{\infty} P[|S_n - n\mu| \geq n\epsilon] < \infty$$

if $EX_1^2 < \infty$. Erdős [4] showed that $EN(\epsilon, \infty) < \infty$ implies that $EX_1^2 < \infty$. We extend the theorem of Hsu and Robbins to some class of dependent random variables. The following lemma will be useful in the sequel considerations.

Lemma . *Let* $\{X_k , k \ge 1\}$ *be a sequence of random variables. Define* $X'_k = X_k I[|X_k| < n\delta]$, $k = 1, \ldots, n$, $\delta > 0$, *and put*

$$S_n = \sum_{k=1}^{n} X_k , \quad S'_n = \sum_{k=1}^{n} X'_k .$$

If

(2) $$ES'_n/n \to 0 , n \to \infty ,$$

then for any given $\epsilon > 0$ *there exists a positive integer* n_0 *such that for* $n \ge n_0$

(3) $$\begin{aligned} P[|S_n| \ge n\epsilon] &\le 4n^{-4}\epsilon^{-4}\Big\{\sum_{j=1}^{n} E(X_j^*)^4 + 4\sum_{j=2}^{n}\sum_{i=1}^{j-1} E(X_j^*)^3 X_i^* \\ &+ 6\sum_{j=2}^{n}\sum_{i=1}^{j-1} E(X_j^*)^2(X_i^*)^2 + 12\sum_{j=1}^{n}\sum_{\substack{i=2\\ i\ne j}}^{j-1}\sum_{\substack{k=1\\ k\ne j}}^{i-1} E(X_j^*)^2 X_i^* X_k^* \\ &+ 4\sum_{j=2}^{n}\sum_{i=1}^{j-1} EX_j^*(X_i^*)^3 + 24\sum_{j=4}^{n}\sum_{i=3}^{j-1}\sum_{k=2}^{i-1}\sum_{l=1}^{k-1} EX_j^* X_i^* X_k^* X_l^*\Big\} \\ &+ \sum_{j=1}^{n} P[|X_j| \ge n\delta] \end{aligned}$$

where $X_m^* = X'_m - EX'_m$, $m \ge 1$.

Proof. Using the Markov's inequality we see that

$$\begin{aligned} P[|S_n| \ge n\epsilon] &\le P[|S_n| \ge n\epsilon , S_n = S'_n] + P[S_n \ne S'_n] \\ &\le P[|S'_n| \le n\epsilon] + \sum_{j=1}^{n} P[|X_j| \le n\delta] \\ &\le P[|S'_n - ES'_n| \ge n\epsilon/2] + P[ES'_n| \ge n\epsilon/2] + \sum_{j=1}^{n} P[|X_j| \ge n\delta] \\ &\le 4n^{-4}\epsilon^{-4}E(S'_n - ES'_n)^4 + \sum_{j=1}^{n} P[|X_j| \ge n\delta] \end{aligned}$$

as by the assumption (2) for $n \ge n_0$ we have $P[|ES'_n| \ge n\epsilon/2] = 0$. Hence we get (3) (cf. [3]).

Corollary 1. *Let* $\{X_k , k \ge 1\}$, *be a sequence of quadruplewise independent random variables satisfying* (2). *Then for any given* $\epsilon > 0$ *there exists a positive* *integer* n_0 *such that for* $n \geq n_0$

(4)
$$P[|S_n| \geq n\epsilon] \leq 4n^{-4}\epsilon^{-4}\left\{\sum_{j=1}^{n} E(X'_j - EX'_j)^4 + 6\sum_{j=2}^{n}\sigma^2 X'_j \sum_{i=1}^{j-1}\sigma^2 X'_i\right\} + \sum_{j=1}^{n} P[|X_j| \geq n\delta] .$$

2. Results. The following theorem states that the theorem of Hsu and Robbins on complete convergence is true for quadruplewise independent random variables.

Theorem 1. *Let* $\{X_k , k \geq 1\}$ *be a sequence of quadruplewise independent identically distributed random variables with* $EX_1 = 0$, $EX_1^2 < \infty$. *Then for any given* $\epsilon > 0$

(5)
$$EN(\epsilon, \infty) = \sum_{n=1}^{\infty} P[|S_n| \geq n\epsilon] < \infty .$$

Proof. From (4) we get for $n \geq n_0$ with $\delta = 1$

$$P[|S_n| \geq n\epsilon] \leq C\left\{n^{-4}\sum_{j=1}^{n} E|X_j|^4 I[|X_j| < n] + n^{-4}\sum_{j=2}^{n} EX_j^2 I[|X_j| < n]\sum_{i=1}^{j-1} EX_i^2 I[|X_i| < n]\right\} + \sum_{j=1}^{n} P[|X_j| \geq n\delta]$$
$$\leq C\left\{n^{-3}E|X_1|^4 I[|X_1| < n] + n^{-2}E|X_1|^2 I[|X_1| < n]\right\} + nP[|X_1| > n],$$

where C is a positive constant depending only on ϵ. It is known that the assumption $EX_1^2 < \infty$ yields:

$$\sum_{n=1}^{\infty} n^{-3}E|X_1|^4 I[|X_1| < n] < \infty ,$$

$$\sum_{n=1}^{\infty} n^{-2}E|X_1|^2 I[|X_1| < n] < \infty ,$$

and

$$\sum_{n=1}^{\infty} nP[|X_1| > n] < \infty$$

(cf.[2]), which proves (5).

Now we need the following concepts (cf.[1] and [8]).

Definition 1. The sequence $\{X_k , k \geq 1\}$, of random varables is called a quadruplewise multiplicative system if

$$EX_{i_1}X_{i_2}X_{i_3}X_{i_4} = 0, \; i_1 < i_2 < i_3 < i_4 , \; i_k \in N, \; k = 1, 2, 3, 4 .$$

Definition 2. The sequence $\{X_k, k \geq 1\}$ of random variables is a quadruplewise strongly multiplicative system if

$$EX_{i_1}^{r_1} X_{i_2}^{r_2} X_{i_3}^{r_3} X_{i_4}^{r_4} = 0, \ i_1 < i_2 < i_3 < i_4 \ , i_k \in N, \ k = 1,2,3,4 \ ,$$

where r_1, r_2, r_3, r_4 can be equal to $0, 1$ or 2 but at least one element of r_1, r_2, r_3, r_4 is equal 1.

Under the above notations we have the following results.

Theorem 2. *Let $\{X_k \ k \geq 1\}$ be a sequence of triplewise independent identically distributed random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. If $X_1' - EX_1', ..., X_n' - EX_n'$ is quadruplewise multiplicative system, then (5) holds true.*

Theorem 3. *Let $\{X_k \ , \ k \geq 1\}$ be a sequence of pairwise independent identically distributed random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. If $X_1' - EX_1', ..., X_n' - EX_n'$ is quadruplewise strongly multiplicative system, then (5) holds true.*

Proofs of Theorems 2 and 3. It is enough to see that under the assumptions of those theorems the inequality (3) reduces to the inequality (4), and next to use the proof of Theorem 1.

For nonidentically distributed random variables we have the following results.

Theorem 4 (cf.[3]). *Let $\{X_k \ , \ k \geq 1\}$ be a sequence of quadruplewise independent random variables. If*

(i) $\sum_{n=1}^{\infty} n^{-4} \sum_{j=1}^{n} E(X_j' - EX_j')^4 < \infty$,

(ii) $\sum_{n=2}^{\infty} n^{-4} \sum_{j=2}^{n} \sigma^2 X_j' \sum_{i=1}^{j-1} \sigma^2 X_i' < \infty$,

(iii) $ES_n'/n \to 0 \ , \ n \to \infty$,

(iv) $\sum_{n=1}^{\infty} \sum_{j=1}^{n} P[|X_j| \geq n\delta] < \infty$,

then (5) holds true.

Proof. The assertion of Theorem 4 is a simple consequence of the inequality (4).

Corollary 2. *Let $\{X_k \ , \ k \geq 1\}$ be a sequence of quadruplewise independent random variables with $EX_k = 0$ and for some $t > 0$, $E|X_k|^{2+t} < \infty$, $k \geq 1$. If*

$$\sum_{n=1}^{\infty} n^{-(2+t)} \sum_{j=1}^{n} E|X_j|^{2+t} < \infty$$

and

$$\sum_{n=2}^{\infty} n^{-4} \sum_{j=2}^{n} \sigma^2 X_j \sum_{i=1}^{j-1} \sigma^2 X_i < \infty ,$$

then the sequence $\{X_k, k \geq 1\}$ satisfies the law of large numbers of Hsu and Robbins.

Proof. We shall verify, that the assumptions (i)-(iv) of Theorem 4 are satisfied. Indeed, we have with $\delta = 1$

$$\sum_{n=1}^{\infty} n^{-4} \sum_{j=1}^{n} E(X_j' - EX_j')^4 \leq 8 \sum_{n=1}^{\infty} n^{-4} \sum_{j=1}^{n} E|X_j'|^4 \leq 8 \sum_{n=1}^{\infty} n^{-(2+t)} \sum_{j=1}^{n} E|X_j|^{2+t} < \infty ,$$

$$|ES_n'/n| = n^{-1} \sum_{j=1}^{n} |EX_j I[|X_j| \geq n]| \leq n^{-(2+t)} \sum_{j=1}^{n} E|X_j|^{2+t} \to 0 , n \to \infty.$$

Corollary 3. *Let $\{X_k , k \geq 1\}$ be a sequence of quadruplewise independent random variables with $EX_k = 0$, $k \geq 1$, and for some $t > 0$, $E|X_k|^{2+t} < L$, $k \geq 1$, where L is a positive constant. Then* (5) *holds true.*

Moreover, one can state the following results.

Theorem 5. *Let $\{X_k , k \geq 1\}$ be a sequence of triplewise independent random variables satisfying* (i)-(iv). *If additionally $X_1' - EX_1' , \ldots, X_n' - EX_n'$, is quadruplewise multiplicative system, then* (5) *holds true.*

Theorem 6. *Let $\{X_k , k \geq 1\}$ be a sequence of pairwise independent random variables satisfying* (i)-(iv). *If additionally $X_1' - EX_1' , \ldots, X_n' - EX_n'$, is quadruplewise strongly multiplicative system, then* (5) *holds true.*

REFERENCES

[1] Alexits, G., *Convergence Problems of Orthogonal Series*, Akadmiai Kiodó, Budapest 1961.

[2] Dugué, D., *Traité de statistique théoretique et appliquée*, Analyse aléatoire-algèbre aléatoire, Masson, Paris 1958.

[3] Duncan, R. and D. Szynal, *A note on the weak and Hsu-Robbins law of large numbers*, Bull. Polish. Acad. Sci. Math. 32 (1984), 729–735.

[4] Erdös, P., *On a theorem of Hsu and Robbins*, Ann. Math. Statist. 20 (1949), 286–291.

[5] Etemadi, N., *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Gebiete 55 (1981), 119–122.

[6] Janson, S., *Some pairwise independent sequences for which the central limit theorem fails*, Uppsala University, Department of Mathematics, Raport 1986, 18 (1986), 1–11.

[7] Hsu, P.L. and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. 33 (1947), 25–31.

[8] Révész, P., *The laws of large numbers*, Academic Press, New York and London 1986.

Instytut Matematyki UMCS
Plac M. Curie Skłodowskiej 1
20-031 Lublin, Poland

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