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Typically Real Functions in Subordination and Majorization

Abstract. This paper deals with the relation between subordination and majorization under the condition for both superordinate functions and majorants to be typically real.

1. Introduction. Let $H(\Omega)$ denote the class of all functions holomorphic in Ω and let

$$\Delta(a,r) = \{z \in \mathbf{C} : |z-a| < r\} \quad , \quad \Delta(r) = \Delta(0,r)\ ,\ \Delta = \Delta(1)\ .$$

Suppose that $f, F \in H(\Delta(r))$. If there is an $\omega \in H(\Delta(r))$ such that $\omega(0) = 0$, $\omega(\Delta(r)) \subset \Delta(r)$ and $f = F \circ \omega$, then we say that f is *subordinate* to F in $\Delta(r)$ or that F is *superordinate* to f in $\Delta(r)$, and we write : $f \prec F$ in $\Delta(r)$.

If now $|f| \le |F|$ in $\Delta(r)$, then we say that f is *majorized* by F or that F is a *majorant* for f in $\Delta(r)$.

It was M.Biernacki [3] who first examined connections between the relations

$$\{f \prec F \text{ in } \Delta(r_1)\} \quad \text{and} \quad \{|f| \le |F| \text{ in } \Delta(r_2)\},$$

under some restrictions imposed on classes in which the functions f and F can vary. If F is equal to the identity mapping, then, according to the Schwarz lemma, the both relations are equivalent.

Let $A \subset H(\Delta)$. For simplicity, we denote the closed convex hull of A by $\overline{co}\,(A)$, and the set of all extreme points of A by $\mathcal{E}A$, and let $c(A)$ be the cone generated by A, i.e. $c(A) = \{\lambda f : \lambda \ge 0\ ,\ f \in A\}$. Next let $N = \{f \in H(\Delta) : f(0) = f'(0) - 1 = 0\}$. Suppose that A, B are subsets of $c(N)$ such that

1° there exist $f_0, g_0 \in A$ and $F_0, G_0 \in B$ for which $f_0 \prec F_0$ and $|g_0| \le |G_0|$ in Δ,

2° there are $r, \rho \in (0,1)$ for that the implications

$$\{f \in A\ ,\ F \in B\ ,\ f \prec F \text{ in } \Delta\} \Longrightarrow \{|f| \le |F| \text{ in } \Delta(r)\} \tag{1}$$

and

$$\{f \in A\ ,\ F \in B\ ,\ |f| \le |F| \text{ in } \Delta\} \Longrightarrow \{f \prec F \text{ in } \Delta(\rho)\} \tag{2}$$

hold. Then the problem is to determine $r_{\mathrm{maj}}(A,B)$, the *radius of majorization* in subordination for the pair (A,B) which is the largest $r \in (0,1]$ such that

(1) holds, and $r_{\mathrm{sub}}(A,B)$, the *radius of subordination* in majorization for the pair (A,B) which is the largest $\rho \in (0,1]$ such that (2) holds. The problem of finding $r_{\mathrm{maj}}(A,B)$ (resp. $r_{\mathrm{sub}}(A,B)$) is sometimes named as the Biernacki (resp. Lewandowski) problem for the pair (A,B). The set

$$D(A,B) = \{z \in \Delta : \{f \in A\,,\ F \in B,\ f \prec F \text{ in } \Delta\} \Longrightarrow |f(z)| \le |F(z)|\}$$

one could call as the set of *majorization* in subordination for the pair (A,B). Clearly,

$$r_{\mathrm{maj}}(A,B) = \max\{r \in (0,1] : \Delta(r) \subseteq D(A,B)\}\,,$$

and if both A and B are invariant under rotations, then $D(A,B)$ is the disk centered at the origin. By definition,

$$r_{\mathrm{maj(sub)}}(A,B) \le r_{\mathrm{maj(sub)}}(A_1,B_1) \text{ and } D(A,B) \subseteq D(A_1,B_1)$$

whenever $A_1 \subseteq A$, $B_1 \subseteq B$.

There are corresponding problems for derivatives. Namely, we may study the implications

$$\{f \in A\,,\ F \in B\,,\ f \prec F \text{ in } \Delta\} \Longrightarrow \{|f'| \le |F'| \text{ in } \Delta(r)\} \tag{3}$$

and

$$\{f \in A\,,\ F \in B\,,\ |f| \le |F| \text{ in } \Delta\} \Longrightarrow \{|f'| \le |F'| \text{ in } \Delta(\rho)\}. \tag{4}$$

Let S denote the familiar class of all univalent functions from N and let

$$S^* = \{f \in S : f(\Delta) \text{ is starlike w.r.t. the origin}\}\,,$$
$$S_{\mathbf{R}} = \{f \in S : f \text{ is real on } (-1,1)\}\quad,\quad S^*_{\mathbf{R}} = S^* \cap S_{\mathbf{R}}$$

and

$$T = \{f \in N : \operatorname{Im} f(z) \operatorname{Im} z \ge 0 \text{ for all } z \in \Delta\}.$$

These compact classes have been thoroughly studied and their basic properties are well known. For instance, $S^* = \{f \in N : \operatorname{Re}\{zf'/f\} > 0 \text{ on } \Delta\}$ and T is identical with $\overline{co}\,(\{q_t : -1 \le t \le 1\}) = \overline{co}\,(S^*_{\mathbf{R}})$, where

$$q_t(z) = z/(1-2tz+z^2) \text{ for } |z|<1\,,\ -1 \le t \le 1. \tag{5}$$

Moreover, $\mathcal{E}T = \{q_t : -1 \le t \le 1\}$ and, according to [13], $\mathcal{E}T = \mathcal{E}S_{\mathbf{R}}$. As to the mentioned problems we have

$$r_{\mathrm{maj}}(c(S),S) \overset{[3]}{=} r_0 \overset{[2]}{=} r_{\mathrm{sub}}(c(S),S), \tag{6}$$

where $r_0 = 0.3908\ldots$ is the unique solution to

$$\log[(1+r)/(1-r)] + 2\arctan r = \pi/2\,,\ 0 < r < 1\,;$$

(7) $$r_{\rm maj}(c(S^*),S^*)\overset{[3]}{=}\sqrt{2}-1\overset{[1]}{=}r_{\rm sub}(c(S^*),S^*)\,;$$

(8) $$r_{\rm maj}(c(N),S)\overset{[18]}{=}(3-\sqrt{5})/2\overset{[7]}{=}r_{\rm maj}(c(N),S^*)\,;$$

(9) $$r_{\rm sub}(c(N),S)\overset{[11]}{=}r_1\overset{[15]}{=}r_{\rm sub}(c(N),S^*)\,,$$

where $r_1 = 0.2955\ldots$ is the unique real solution of the equation $r^3+r^2+3r=1$;

(10) $$\{f\in c(N)\,,\ F\in S\,,\ f\prec F \text{ in } \Delta\}\Longrightarrow\{|f'|\le|F'| \text{ in } \Delta(3-\sqrt{8})\}\,;$$

the radius $3-\sqrt{8}$ is best possible and cannot be increased even then if we reduce the classes $c(N)$ and S to $c(S^*)$ and S^*, respectively, see [19, 7, 8];

(11) $$\{f\in H(\Delta)\,,\ F\in S\,,\ |f|\le|F| \text{ in } \Delta\}\}\Longrightarrow\{|f'|\le|F'| \text{ in } \Delta(2-\sqrt{3})\}\,;$$

the radius $2-\sqrt{3}$ is best possible and cannot be increased even then if we reduce the classes $H(\Delta)$ and S to $c(S^*)$ and S^*, respectively, see [16, 17].

For details and a large list of similar results see [5, 9]. Golusin [6] found that the maximal domain of univalence for the class T is the lens-shaped set $\Delta(-i,\sqrt{2})\cap\Delta(i,\sqrt{2})$ and hence the radius of univalence for the class T is equal to $\sqrt{2}-1$. Moreover Kirwan [12] proved that the same number is the radius of starlikeness in S. Hence the implications (1) – (4) are sensible for $B=T$ and $A=c(T)$ or $A=c(N)$. Unfortunately, these results following (7) – (11) are not sharp.

The main sharp theorem concerns the explicit description of the set $D(c(T),T)$, see Theorem 1. From this we shall deduce that $r_{\rm maj}(c(T),T)=0.3637\ldots$, see Theorem 2. The proof of Theorem 1 is based on an integral representation for bounded typically real functions [20–23] and a detailed description of the sets $\{zf'(z)/f(z)\,;\ f\in T\}$, $z\in\Delta$ [14, 22–23]. Finally, we shall show that (4) holds with $A=H(\Delta)$, $B=T$ and $\rho=2-\sqrt{3}$, and that the radius $2-\sqrt{3}$ cannot be increased, see Theorem 3.

2. Elementary observations. From (7) – (11) it follows

Proposition 1.

(i) $r_{\rm maj}(c(S^*_{\mathbf R}),S^*_{\mathbf R})=r_{\rm sub}(c(S^*_{\mathbf R}),S^*_{\mathbf R})=\sqrt{2}-1$.

(ii) $r_{\rm maj}(c(N),S^*_{\mathbf R})=(3-\sqrt{5})/2$.

(iii) $r_{\rm sub}(c(N),S^*_{\mathbf R})=r_1$, *where* r_1 *is defined in* (9).

(iv) $\{f\in c(S^*_{\mathbf R})\,,\ F\in S^*_{\mathbf R}\,,\ f\prec F \text{ in } \Delta\}\Longrightarrow\{|f'|\le|F'| \text{ in } \Delta(3-\sqrt{8})\}$, *and the radius* $3-\sqrt{8}$ *is best possible.*

(v) $\{f\in c(S^*_{\mathbf R})\,,\ F\in S^*_{\mathbf R}\,,\ |f|\le|F| \text{ in } \Delta\}\Longrightarrow\{|f'|\le|F'| \text{ in } \Delta(2-\sqrt{3})\}$, *and the radius* $2-\sqrt{3}$ *is best possible.*

Proof. On account of (7) – (11) it is sufficient to consider such pair of holomorphic functions with real coefficients which show that the results (7) – (11) are sharp. Namely, like in [1, 3, 8, 16] let us examine the functions $f_\varepsilon=(1-\varepsilon)q_{2\varepsilon-1}$ for $0\le\varepsilon<1$ and $g_{\varepsilon,\lambda}(z)\equiv\{(1-\varepsilon)[1+|\lambda|(1+\varepsilon)z]/[1+|\lambda|(1-\varepsilon)z]\}z/(1-\lambda z)^2$ for $-1<\lambda<1$ and $0\le\varepsilon<(1-|\lambda|)^4$. Clearly, for all $0\le\varepsilon<1$ we have $f_\varepsilon\in c(S^*_{\mathbf R})$ and $f_\varepsilon\prec f_0$ in Δ. If $\sqrt{2}-1<r<1$, then $d(|f_\varepsilon(ir)|^2)/d\varepsilon>0$ at the point $\varepsilon=0$, i.e.

$r_{maj}(c(S^*_{\mathbf{R}}), S^*_{\mathbf{R}}) \le \sqrt{2}-1$. If now $3-\sqrt{8} < r < 1$, then $df'_\varepsilon(r)/d\varepsilon > 0$ at the point $\varepsilon = 0$, i.e. (iv) holds. Next observe that for all $-1 < \lambda < 1$ and $0 \le \varepsilon < (1-|\lambda|)^4$ we have $g_{\varepsilon,\lambda} \in c(S^*_{\mathbf{R}})$ and $|g_{\varepsilon,\lambda}| \le |g_{0,\lambda}|$ in Δ. If $\sqrt{2}-1 < r < \lambda\rho < \rho$ and $a = ir/\lambda$, then

$$\lim_{\varepsilon \to 0^+} \mathrm{Re}\{[g_{\varepsilon,\lambda}(a) - g_{0,\lambda}(a)]/[\varepsilon a g'_{0,\lambda}(a)]\} = -\mathrm{Re}\{(1-ri)^2/(1+ri)^2\} > 0 \,,$$

so the subordination $g_{\varepsilon,\lambda} \prec g_{0,\lambda}$ in $\Delta(\rho)$ fails to be true, i.e. $r_{sub}(c(S^*_{\mathbf{R}}), S^*_{\mathbf{R}}) \le \sqrt{2}-1$. If now $2-\sqrt{3} < -\lambda\rho < \rho$, then $d[g'_{\varepsilon,\lambda}(\rho) - g'_{0,\lambda}(\rho)]/d\varepsilon > 0$ at the point $\varepsilon = 0$, i.e. (v) holds.

Golusin [7] observed that $q_1(x^2) \le |q_1(x)|$ for $-\rho < x < 0$ implies that $\rho \le (3-\sqrt{5})/2$, whence $r_{maj}(c(N), S^*_{\mathbf{R}}) \le (3-\sqrt{5})/2$. Finally, Lewandowski [15] noticed that the inequality $r_1 < \rho < 1$ leads to $f(-\rho) > q_{-1}(\rho)$, where $f(z) \equiv zq_{-1}(z)$. Since $q_{-1}(\Delta(\rho)) \cap \mathbf{R} = (q_{-1}(-\rho), q_{-1}(\rho))$, the subordination $f \prec q_{-1}$ in $\Delta(\rho)$ does not hold, i.e. $r_{sub}(c(N), S^*_{\mathbf{R}}) \le r_1$.

Proposition 2.
(i) $(\sqrt{2}-1)^2 \le r_{maj}(c(T), T) \le \sqrt{2}-1$,
(ii) $(\sqrt{2}-1)(3-\sqrt{5})/2 \le r_{maj}(c(N), T) \le (3-\sqrt{5})/2$,
(iii) $(\sqrt{2}-1)^2 \le r_{sub}(c(T), T) \le \sqrt{2}-1$,
(iv) $0.1224\cdots = (\sqrt{2}-1)r_1 \le r_{sub}(c(N), T) \le r_1$, *where* r_1 *is defined in* (9).
(v) $\{f \in c(N)\,,\ F \in T\,,\ f \prec F$ *in* $\Delta\} \Longrightarrow \{|f'| \le |F'|$ *in* $\Delta((\sqrt{2}-1)^3)\}$.
The best possible radius is no larger than $(\sqrt{2}-1)^2$.
(vi) $\{f \in H(\Delta)\,,\ F \in T\,,\ |f| \le |F|$ *in* $\Delta\} \Longrightarrow \{|f'| \le |F'|$ *in* $\Delta((\sqrt{2}-1)(2-\sqrt{3}))$.
The best possible radius is no larger than $2-\sqrt{3}$.

Proof. Since $S^*_{\mathbf{R}} \subset T$, all the upper bounds result from Proposition 1. The estimation from bellow we motivate as follows. For any $g \in H(\Delta(\rho))$ and $r > 0$ consider the new function $g_r(z) \equiv g(rz)/r$ which is in $H(\Delta(\rho/r))$. Hence for every $r > 0$ the condition "$f \prec F$ in $\Delta(\rho)$" is equivalent to "$f_r \prec F_r$ in $\Delta(\rho/r)$". Indeed, if $f = F \circ \omega$ and $|\omega(z)| \le |z|$ for $|z| < \rho$, then $f_r = F_r \circ \omega_r$ with $|\omega_r(z)| = |\omega(rz)|/r \le |z|$ for $|z| < \rho/r$, and conversely. Similarly, for every $r > 0$ the condition "$|f| \le |F|$ in $\Delta(\rho)$" means "$|f_r| \le |F_r|$ in $\Delta(\rho/r)$". By the Kirwan result [12] we have that $F_{\sqrt{2}-1} \in S^*$ and $f_{\sqrt{2}-1} \in c(S^*)$ whenever $F \in T$ and $f \in c(T)$. Thus all the lower bounds are simple consequence of the facts (7) – (11).

3. Auxiliary lemmas. Let $\mathbf{P}(a,b)$ denote the set of all probability measures on the compact line segment $[a,b]$ and let δ_x mean the Dirac measure at the point x. The lemmas below will be used to obtain Theorems 1–3. Nevertheless they are interesting in themselves. The first result concerns nonvanishing typically real functions, and hence bounded typically real functions. Let us recall that for the class T we have

$$T = \left\{ \int_{-1}^{1} q_t \, d\nu(t) : \nu \in \mathbf{P}(-1,1) \right\} , \tag{12}$$

the Robertson representation.

Lemma 1 [20–23]. *The class*

$$T_0 = \{f \in H(\Delta) : f(0) = 1 \,,\ 0 \in \mathbf{C} \setminus f(\Delta)\,,\ \mathrm{Im}\, f(z) \mathrm{Im}\, z \ge 0 \text{ on } \Delta\}$$

is identical with the set $\{f/q_{-1} : f \in T\}$. *Hence,* $\omega \in H(\Delta)$ *with* $\omega(0) = 0, |\omega(z)| < 1$ *and* $\operatorname{Im}\omega(z)\operatorname{Im} z \geq 0$ *on* Δ *if and only if* $\omega \in H(\Delta)$ *and* $(1+\omega)^2 q_{-1}/(1-\omega)^2 \in T$.

Remarks. The proof of Lemma 1 one can find also in [10]. We let add that $T_0 = \overline{co}\,(S_0)^+_{\mathbf{R}}$, where the class $(S_0)_{\mathbf{R}}$ consists of all nonvanishing univalent functions $f \in H(\Delta)$ real on $(-1,1)$ and normalized by $f(0) = 1$, and where $(S_0)^+_{\mathbf{R}} = \{f \in (S_0)_{\mathbf{R}} : f'(0) > 0\}$. By Lemma 1 we have $\mathcal{E}T_0 = \{q_t/q_{-1} : -1 \leq t \leq 1\}$. Furthermore, $\mathcal{E}(S_0)^+_{\mathbf{R}} = \{q_t/q_{-1} : -1 < t \leq 1\}$ and

$$\mathcal{E}(S_0)_{\mathbf{R}} = \{q_t/q_{-1} : -1 < t \leq 1\} \cup \{q_t/q_1 : -1 \leq t < 1\}\,,$$

see [13], and $\mathcal{E}(S_0)_{\mathbf{R}} = \sigma(S_0)_{\mathbf{R}}$, the set of all support points of the class $(S_0)_{\mathbf{R}}$, see [10]. Like in the theorem 4.3 [10] we can get that

$$\sigma(S_0)^+_{\mathbf{R}} = \{(1-\lambda)q_s/q_{-1} + \lambda : 0 \leq \lambda < 1\,,\ -1 < s \leq 1\}\,.$$

It suffices to consider the functionals

$$L_s(f) = -6(1+4s^2)f'(0) + 12sf''(0) - f'''(0)\,,\ -1 < s \leq 1\,,$$

that assume theirs maxima over $(S_0)^+_{\mathbf{R}}$ at the functions $g_{s,\lambda} = (1-\lambda)q_s/q_{-1} + \lambda \in (S_0)^+_{\mathbf{R}}$, respectively, where $0 \leq \lambda < 1$. In fact, for all $t \in [-1,1]$ we have

$$L_s(q_t/q_{-1}) = 2(1+t)L_s(q_t) = -48(1+t)(t-s)^2 \leq 0 = L_s(g_{s,\lambda})$$

and Re L_s is not constant on $(S_0)^+_{\mathbf{R}}$. Thus, for $-1 < s \leq 1$ and $0 \leq \lambda < 1$ the functions $g_{s,\lambda} \in \sigma(S_0)^+_{\mathbf{R}}$.

As a corollary to Lemma 1 we get

Lemma 2 [20, 22–23]. *Let* $F \in T$. *Then*

$$f \in c(T)\,,\ f \prec F \text{ in } \Delta$$

if and only if there is $\mu \in \mathbf{P}(-1,1)$ *such that* $f = F \circ \omega_\mu$, *where*

$$q_1(\omega_\mu(z)) \equiv \int_{-1}^{1} [(1+t)q_t(z)/2]\, d\mu(t)\,. \tag{13}$$

The next result concerns quotients of some integrals.

Lemma 3 [14, 22–23]. *Let* $w < 1$ *or* $\operatorname{Im} w \neq 0$, *let* $a = 1$, $b = 1/(1-w)$, $d = \overline{w}/(\overline{w} - w)$, $r = |d|$, *and let*

$$\varphi(w,\mu) = \int_0^1 (1-tw)^{-2}\, d\mu(t) \Big/ \int_0^1 (1-tw)^{-1}\, d\mu(t)\,,\quad \mu \in \mathbf{P}(0,1)\,. \tag{14}$$

The set $D_w = \{\varphi(w,\mu) : \mu \in \mathbf{P}(0,1)\}$ *is a compact convex circular region. More precisely,*

(i) *If* $w < 1$, *then* D_w *is the line segment joining* a *and* b.

(ii) *If* $\operatorname{Re} w \le 1$, $\operatorname{Im} w \ne 0$, *then* $D_w = \overline{\Delta}(d,r) \cap \overline{\Delta}(a+b-d,r)$, *i.e.* $\partial D_w = C \cup C^*$, *where* $C = \{\varphi(w,\delta_\lambda) : 0 \le \lambda \le 1\}$ *and* C^* *is the reflection of* C *in the point* $(a+b)/2$. *In particular, for* $\operatorname{Re} w = 1$, $\operatorname{Im} w \ne 0$ *we have* $D_w = \overline{\Delta}(d,r)$.

(iii) *For the case* $\operatorname{Re} w > 1$, $\operatorname{Im} w \ne 0$ *see* [14, 22–23].

In [14], it was described the set $\{zf'(z)/f(z) : f \in T\}$ for every $z \in \Delta$. Its boundary, except for real z, consists of at most four circular arcs. In particular, it was proved

Lemma 4 [14]. *For* $|z| \le 2-\sqrt{3}$ *and* $F \in T$, *we have the following sharp estimation*

$$|zF'(z)/F(z)| \ge (1-|z|)/(1+|z|)\ .$$

The radius $2-\sqrt{3}$ *is best possible.*

Now we deduce a characterization of the set $D(c(T),T)$.

Lemma 5 [22, 23]. $D(c(T),T) = D \cap (-D)$, *where*

$$D = \{z \in \Delta : D_{w(z)} \subset \{\zeta : \operatorname{Re}\zeta \ge 0\}\}\ ,\ w(z) \equiv 4z/(1+z)^2$$

and D_w *is defined in* Lemma 3.

For the convenience of the reader (items [22–23] are in Polish), we give

Proof. Observe first that $(-1,1) \subset D(c(T),T) \cap D \cap (-D)$ as functions from $c(T)$ are increasing on $(-1,1)$, and for $-1 < x < 1$ the set $D_{w(x)}$ is the closed line segment with ends 1 and $(1+x)^2/(1-x)^2$. According to Lemma 2, $z \in D(c(T),T) \setminus \mathbf{R}$ if and only if $z \in \Delta \setminus \mathbf{R}$ and $|F(\omega_\mu(z))| \le |F(z)|$ for all $F \in T$ and $\mu \in \mathbf{P}(-1,1)$, where ω_μ is defined in (13). By the maximum principle, $z \in D(c(T),T) \setminus \mathbf{R}$ if and only if $z \in \Delta \setminus \mathbf{R}$ and $|F(\zeta)| \le |F(z)|$ for all $F \in T$ and $\zeta \in \partial\{\omega_\mu(z) : \mu \in \mathbf{P}(-1,1)\}$. However, from Lemma 1 or 2 it follows that for each $z \in \Delta \setminus \mathbf{R}$ the set $\{[1+\omega_\mu(z)]^2/[1-\omega_\mu(z)]^2 : \mu \in \mathbf{P}(-1,1)\}$ is the closed convex hull of the circular arc $[-1,1] \ni t \mapsto (q_t/q_{-1})(z) = 1 + 2(1+t)q_t(z)$, i.e.

$$\partial\{\omega_\mu(z) : \mu \in \mathbf{P}(-1,1)\} = \{\omega(z,t) : -1 \le t \le 1\} \cup \{-\omega(-z,t) : -1 \le t \le 1\}\ ,$$

where

$$\omega(\zeta,t) = q_1^{-1}((1+t)q_t(\zeta)/2) \text{ for } |\zeta| < 1\ ,\ -1 \le t \le 1\ . \tag{15}$$

So, $z \in D(c(T),T) \setminus \mathbf{R}$ if and only if $z \in \Delta \setminus \mathbf{R}$ and $\max\{|F(\omega(z,t))|\ ,\ |F(-\omega(-z,t))|\} \le |F(z)|$ for all $F \in T$ and $-1 \le t \le 1$. Since $F \in T$ whenever $\zeta \mapsto -F(-\zeta)$ is in T, we get that $D(c(T),T) = \widetilde{D} \cap (-\widetilde{D})$, where

$$\widetilde{D} = \{z \in \Delta : |F(\omega(z,t))| \le |F(z)| \text{ for all } F \in T \text{ and } -1 \le t \le 1\}\ .$$

We want to show that $\widetilde{D} = D$. Let $F \in T$. By (12) there is a $\nu \in \mathbf{P}(0,1)$ such that $F = \int_0^1 q_{2s-1} d\nu(s)$, and from (15) it follows that $\omega(\zeta,t)+1/\omega(\zeta,t) \equiv 2(\zeta+1/\zeta+1-t)/(1+t)$, i.e. $q_{2s-1}(\omega(\zeta,t)) \equiv (1+t)q_{s(1+t)-1}(\zeta)/2$. Hence

$$F(\omega(z,t)) = (\lambda w/4)\int_0^1 (1-\lambda s w)^{-1} d\nu(s) \text{ , where } 2\lambda = 1+t \text{ and } w = w(z) .$$

Thus

$$\widetilde{D} = \{z \in \Delta : q_{\nu,z}(\lambda) \le q_{\nu,z}(1) \text{ for } \nu \in \mathbf{P}(0,1) \text{ and } 0 \le \lambda \le 1\},$$

where we have denoted

$$q_{\nu,z}(\lambda) = \left|\int_0^1 \lambda(1-\lambda s w)^{-1} d\nu(s)\right|^2 \text{ and } w = w(z) .$$

Next observe that the condition

$$(16) \qquad q_{\nu,z}(\lambda) \le q_{\nu,z}(1) \text{ for all } \nu \in \mathbf{P}(0,1) \text{ and } 0 \le \lambda \le 1$$

is equivalent to

$$(17) \qquad q'_{\nu,z}(1) \ge 0 \text{ for all } \nu \in \mathbf{P}(0,1) .$$

Indeed, the implication (16)$\Longrightarrow$(17) is trivial. Now, suppose that (17) holds and let $\nu \in \mathbf{P}(0,1)$, $0 < \lambda \le 1$ and $h(s) = \lambda s$ for $0 \le s \le 1$. Then $\widetilde{\nu} = \nu \circ h^{-1} \in \mathbf{P}(0,1)$, $\widetilde{\nu}((\lambda,1]) = 0$ and

$$0 \le q'_{\widetilde{\nu},z}(1) = 2\mathrm{Re}\left\{\int_0^\lambda (1-\tau w)^{-2} d\widetilde{\nu}(\tau) \int_0^\lambda (1-\tau\overline{w})^{-1} d\widetilde{\nu}(\tau)\right\}$$

$$= 2\mathrm{Re}\left\{\int_0^1 (1-s\lambda w)^{-2} d\nu(s) \int_0^1 (1-s\lambda\overline{w})^{-1} d\nu(s)\right\} = q'_{\nu,z}(\lambda)/\lambda .$$

Since $\lambda \in (0,1]$ and $\nu \in \mathbf{P}(0,1)$ were arbitrary, the functions $\lambda \mapsto q_{\nu,z}(\lambda)$ increase on $[0,1]$, i.e. (16) holds. Thus (16) and (17) are equivalent and hence $\widetilde{D} = D$ because $\mathrm{Re}\,\varphi(w,\nu) = q'_{\nu,z}(1)/(2q_{\nu,z}(1))$. The proof is complete.

4. Main results.

Theorem 1 [22–23].
(i) *The set $D(c(T),T)$ is symmetric about the coordinate axes and starlike with respect to the origin.* (ii) *The set $D(c(T),T) \cup \{-1,1\}$ is compact and its boundary is the union of Jordan arcs Γ_1, Γ_2 with common ends: $\pm i\tan(t_0/2)$, where*

$$\Gamma_j = \{z : (1+z)/(1-z) = \rho_j(t)e^{it} \text{ , } \pi/8 \le |t| \le t_0\} \text{ , } j = 1,2,$$

$$\rho_1(t) = \sqrt{\cos(2t)}/(\sqrt{2}|\sin(2t)| - 1) \text{ , } \rho_2(t) \equiv 1/\rho_1(t)$$

and $t_0 = 0.7064\ldots$ *is the unique solution of the equation:* $\rho_1(t) = 1, \pi/8 < t < \pi/4$.

Again, since items [22–23] are in Polish, we let the reader to know

Proof. (i). Let f_r mean the function $z \mapsto f(rz)/r$, where $f \in H(\Delta)$ and $r \in (0,1) \cup \{-1\}$. If $f \in c(T)$, $F \in T$ and $f \prec F$ in Δ, then $f_r \in c(T)$, $F_r \in T$ and $f_r \prec F_r$ in Δ for all $r \in (0,1) \cup \{-1\}$. Thus, if $z \in D(c(T),T)$, then also $\bar{z} \in D(c(T),T)$ and $rz \in D(c(T),T)$ for all $0 < r < 1$ and $r = -1$.

(ii). Apply Lemmas 5 and 3. Then $D(c(T),T) = D \cap (-D)$ and the image of D by means of the function $1 + 4q_1$ is the set

$$\Omega = \{1/(1-w) : w \in \mathbf{C} \setminus [1,+\infty) \text{ and } D_w \subset \{\zeta : \operatorname{Re}\zeta \geq 0\}\},$$

where D_w is determined in Lemma 3. The inequality Re $\varphi(w,\delta_t) \geq 0$ for $0 \leq t \leq 1$ implies that Re $w \leq 1$, so in the case Im $w \neq 0$ the boundary arcs of D_w have equations:

$$[0,1] \ni t \mapsto 1/(1-tw)\ ,\ [0,1] \ni \lambda \mapsto [1-\lambda+\lambda/(1-w)^2]/[1-\lambda+\lambda/(1-w)]\ .$$

Since Re$w \leq 1$, the first arc lies in the closed right halfplane. Imposing on the second arc to be in the closed right halfplane we get that

$$\Omega = \{u+iv : u \geq 0\ ,\ v \in \mathbf{R} \text{ and } |v| \leq \sqrt{2u} + \sqrt{u(1+u)}\}$$

In fact,

$$\Omega = \{u+iv : u \geq 0 \text{ and } p(u,v,\lambda) \geq 0 \text{ for } 0 \leq \lambda \leq 1\}\ ,$$

where $p(u,v,\lambda) \equiv \lambda^2[(u-1)^2+v^2](u+1)+\lambda[(u-1)(u+2)-v^2]+1$. Since $p(u,v,0) = 1$ and $p(u,v,1) = u(u^2+v^2) \geq 0$, we have $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\begin{aligned}\Omega_1 &= \{u+iv : u \geq 0 \text{ and } \lambda_{u,v} \geq 1\}\ ,\\ \Omega_2 &= \{u+iv : u \geq 0 \text{ and } \lambda_{u,v} \leq 0\}\ ,\\ \Omega_3 &= \{u+iv : 0 \leq \lambda_{u,v} \leq 1 \text{ and } p(u,v,\lambda_{u,v}) \geq 0\}\end{aligned}$$

and $p'_\lambda(u,v,\lambda_{u,v}) = 0$. After easy calculations we obtain that

$$\Omega_1 \cup \{u+iv \in \Omega_3 : 0 \leq u \leq 1\} = \{u+iv : 0 \leq u \leq 1\ ,\ |v| \leq \sqrt{2u} + \sqrt{u(1+u)}\}$$

and

$$\Omega_2 \cup \{u+iv \in \Omega_3 : u \geq 1\} = \{u+iv : u \geq 1\ ,\ |v| \leq \sqrt{2u} + \sqrt{u(1+u)}\}\ .$$

Thus

$$\Omega = \{\rho e^{it} : |t| \leq \pi/4\ ,\ \rho \geq 0\} \cup \{\rho e^{it} : \pi/4 \leq |t| \leq \pi/2\ ,\ 0 \leq \rho \leq \rho_1^2(t/2)\}$$

and hence

$$\begin{aligned}D = &\{(\rho e^{it}-1)/(\rho e^{it}+1) : |t| \leq \pi/8\ ,\ 0 \leq \rho < \infty\} \cup \\ &\cup \{(\rho e^{it}-1)/(\rho e^{it}+1) : \pi/8 < |t| \leq \pi/4\ ,\ 0 \leq \rho \leq \rho_1(t)\}\ .\end{aligned}$$

By Lemma 5, the proof is complete.

Remarks. Using a computer one easily checks that the set $D(c(T),T)$ is convex. Unfortunately, it seems that a direct proof of the fact that the curve $\Gamma_1 \cup \Gamma_2$ is convex can involve some heavy calculations. We let add that by Theorem 1 the following proper inclusions hold

$$D(\pi/8) \subset D(c(T),T) \subset D(t_0) \cup \{-i\tan(t_0/2)\ ,\ i\tan(t_0/2)\}\ ,$$

where we have denoted

$$D(\alpha) = \{z \in \mathbf{C} : |\arg[(1+z)/(1-z)]| < \alpha\} = \\ = \Delta(-i\cot\alpha, 1/\sin\alpha) \cap \Delta(i\cot\alpha, 1/\sin\alpha)\ .$$

Theorem 2 [22–23].

$$r_{\mathrm{maj}}(c(T),T) = [(13 - 2\sqrt{9+5\sqrt{10}})/(13+2\sqrt{9+5\sqrt{10}})]^{1/2} = 0.3637\ldots$$

.

Proof. Putting $\tan t = x\sqrt{2}$ we find the minimum of the function

$$t \mapsto |(\rho_1(t)e^{it} - 1)/(\rho_1(t)e^{it} + 1)|^2 = p(t)$$

in the interval $(\pi/8, \pi/4)$. To this end, note that $p'(t)$ has the same sign as the polynomial $x \mapsto x(2x^2+1)(6x^2-8x+3)(2x^2+4x-3)$ and that the minimum of p is assumed at the point $t = \operatorname{arc\,tan}(\sqrt{5}-\sqrt{2}) = 0.6879\ldots$.

Theorem 3. $\{f \in H(\Delta)\ ,\ F \in T\ ,\ |f| \le |F|\ \mathit{in}\ \Delta\} \Longrightarrow \{|f'| \le |F'|\ \mathit{in}\ \Delta(2-\sqrt{3})\}$ *and the number* $2-\sqrt{3}$ *is best possible.*

Proof. Because of Lemma 4, the proof is the same as in [16, 17].

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